TORIC VARIETIES: A SHORT INTRODUCTION

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1. Introduction

Toric varieties are a very important class of algebraic varieties. They provide an alternative way to see many phenomena in algebraic geometry. Whole courses are sometimes devoted to the study of these mathematical objects. In this text however, we simply give the bare minimum needed to define such varieties.

We shall begin by examining a familiar example of a toric variety to motivate the later constructions. We will then see that toric varieties are constructed by "gluing" together affine varieties which correspond to a specific kind of geometric objects — a type of cone. In turn, these cones are defined formally (but by no means in full generality). Next, some of their useful properties are discussed. The following section discusses the semigroup S_σ constructed from the cone σ. S_σ will then be used to define a $\mathbb{C}$-algebra which will be the affine ring corresponding to a cone σ. Toward the end the section, the gluing the varieties given by these affine rings is discussed. Finally, a few descriptive example are presented in decreasing detail to illustrate the construction.

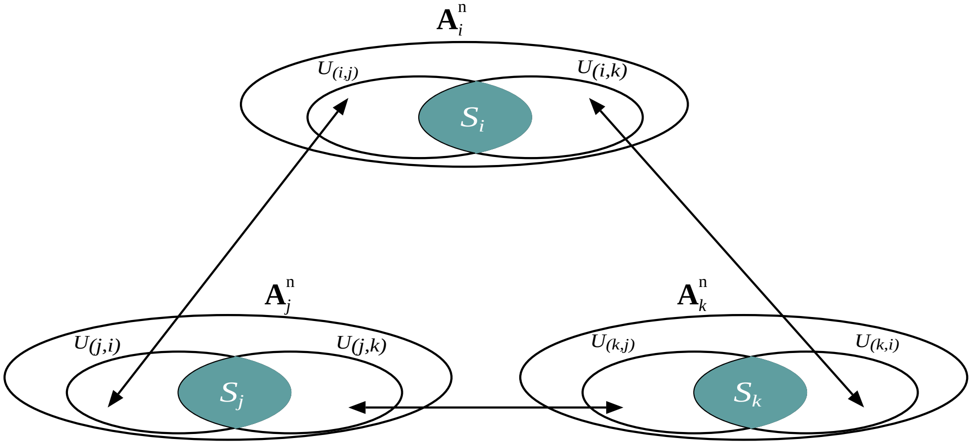

Figure 1. Three copies of $\mathbb{A}^n$ that are contained in $\mathbb{P}^n$.

2. A Familiar Case

We know that projective space $\mathbb{P}^n$ can be seen as a bunch ($n+1$ to be precise) of affine spaces of dimension n glued together. Indeed, taking $V_i = \{(a_0 : \cdots : a_n) \mid a_i \neq 0\}$ and observing that $V_i \cong \mathbb{A}^n$, suggest how this can be done. Further, the sets V_i and V_j have a nontrivial intersection, namely

$$V_i \cap V_j = \{(a_0 : \cdots : a_n) \mid a_i \neq 0, a_j \neq 0\}\,.$$

On the other hand, if $i \neq j$ then $V_i \neq V_j$. Hence, if we denote V_i by $\mathbb{A}^n_i$ and the think of $V_i \cap V_j$ as sitting inside $\mathbb{A}^n_i$ by calling it $U_{(i,j)}$, we get a situation as depicted by Figure 1. Thus, we require that $U_{(i,j)}$ and $U_{(j,i)}$ be identified somehow (via an isomorphism), but this must be done so that S_i, S_j, S_k remain isomorphic. The latter requirement comes from the observation that all three shaded regions correspond to $V_i \cap V_j \cap V_k$ in $\mathbb{P}^n$.

This idea can be generalized. Instead of copies of $\mathbb{A}^n$ why not take any collection of affine varieties and glue them together so that similar requirements are satisfied? In the next sections we shall describe a way to construct such affine varieties, so that the notion of gluing makes sense.

3. Fans and Cones

In order to define the gluing process in a consistent way, we need to have a good handle on the affine varieties that will be used in the procedure. Here, we develop the notion of a fan of cones. Cones are geometric objects that we shall use to define the affine varieties. The fan will be a collection of cones that satisfy a natural notion of “gluing.” This will subsequently be extended to what we are really after — gluing affine varieties.

3.1. Cones.

Consider a lattice N that is isomorphic to $\mathbb{Z}^n$. Denote by $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ the real vector space induced by N.[1] Take $\{v_1, \ldots, v_l\} \subseteq N$ and let

$$\sigma = \langle v_1, \ldots, v_l \rangle = \left\{ v \in N_{\mathbb{R}} \;\middle|\; \sum_{i=1}^{l} r_i v_i, \text{ for } r_i \geq 0 \right\}$$

where σ is called a *rational polyhedral cone.* This is almost the type of cone that we shall be working with. We need *strongly convex* rational polyhedral cones. In other words, we shall require that σ above is such that if $v \in \sigma$ then $-v \notin \sigma$, i.e. σ contains no line through the origin. From now on, when we say *cone* we shall mean a strongly convex rational polyhedral cone.

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[1] If $N = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$, then $N_{\mathbb{R}} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n$.

3.2. **Dual Cones, Faces and Fans.** Denote by $M = \mathrm{Hom}(N, \mathbb{Z})$ the dual lattice of N. If σ is a cone, then the *dual cone* $\sigma^\vee$ is given by

$$\sigma^\vee = \{u \in M_\mathbb{R} \mid \forall v \in \sigma, u(v) \geq 0\}\,.$$

Note that $\sigma^\vee$ is also a convex cone if σ is.[2] In fact, one give a *dual definition* for a convex cone, namely

$$\sigma = \{v \mid u_i(v) \geq 0\}$$

where the u_i are generators of $\sigma^\vee$. For a proof, see [Fulton, p. 12].

Also, we have $u^\perp = \{v \in N_\mathbb{R} \mid u(v) = 0\}$ for any dual vector $u \in M_\mathbb{R}$. We define a *face* τ of the cone σ by taking $u \in \sigma^\vee \cap M$ and putting

$$\tau = \sigma \cap u^\perp = \{v \in \sigma \mid u(v) = 0\}\,.$$

Finally, a fan Δ is a collection of cones which satisfy properties similar to those for a simplicial complex:

F1 if $\tau \subseteq \sigma$ is a face of a cone $\sigma \in \Delta$, then τ is a cone and $\tau \in \Delta$;
F2 $\sigma, \sigma' \in \Delta$ then $\sigma \cap \sigma'$ is a face of both σ and σ'.

Note that **F2** gives a "natural gluing" of cones; that is the face that two cones share can be thought of as a gluing of one cone to the other.

Here is an interesting property of the shared face of two cones.

Proposition 1. *Given two cones $\sigma, \sigma' \in \Delta$, let the face $\tau = \sigma \cap \sigma'$. Then there is $u \in \sigma^\vee \cap (-\sigma')^\vee$ such that*

$$\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp.$$

Proof. Take the cone $\gamma = \sigma \cup (-\sigma')$. Note that γ may not be strongly convex, since if $v \in \sigma$ and $-v \in -\sigma'$ we have a line through the origin. However, it contains both σ and $-\sigma'$. Consider $u \in \gamma^\vee$ — thus $u \in \sigma^\vee \cap (-\sigma')^\vee$. Take $u = u_1 + \cdots + u_n$ where n is the dimension of $\sigma^\vee$ and the u_i are independent generators. Hence, $v \in \gamma$ is such that $u(v) = 0$ iff $u(-v) = 0$. Also, if $v \in \gamma$ but $-v \notin \gamma$, then at least one of the u_i is non zero on v (otherwise $u_i(v) = u_i(-v) = 0$ so $-v \in \gamma$ by the dual definition of σ.). So

$$\gamma \cap u^\perp = \gamma \cap (-\gamma) = (\sigma \cup (-\sigma')) \cap (\sigma' \cup (-\sigma))$$

by definition of $u^\perp$.

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[2] This fact is not essential to the exposition, so we shall not prove it. However, we shall need the fact the dual cone is finitely generated, so here is a short outline of the idea. First, we show that we can find generators for $\sigma^\vee$ where $u \in \sigma^\vee$ is a generator if it annihilates some subset of size $n-1$ of the generators for σ (where n is the number of independent generators of σ). In particular, taking only one such u per subset (independent of the previous ones) means that we need only finitely many such vectors. Then the claim follows by observing that no coefficient in a linear combination of these u's (which are assumed independent) may be negative for this would not give a vector in $\sigma^\vee$. For more information see [Fulton, p. 11 (Farkas' Theorem)].

Now, if $v \in \sigma \cap u^{\perp}$, then $v \in \sigma' \cup (-\sigma)$ by the above, so we can write $v = w' - w$ for $w' \in \sigma'$ and $w \in \sigma$. Thus, $v + w \in \sigma$ and $v + w \in \sigma'$, i.e. $v + w \in \tau$. But then $v \in \tau$ since if $\tau = \sigma \cap (u')^{\perp}$, for $u' \in \sigma^{\vee}$, then $0 = u'(v + w) = u'(v) + u'(w)$ and both $u'(v) \geq 0$ and $u'(w) \geq 0$, so $u'(v) = 0$.

Hence, $\tau = \sigma \cap u^{\vee}$. Similarly, but using $-u$, we show that $\tau = \sigma' \cap u^{\vee}$ □

4. Toric Varieties

4.1. **The Semigroup S_σ.** Take a cone σ and define S_σ by

$$S_\sigma = \sigma^{\vee} \cap M = \{u \in M \mid \forall v \in \sigma, u(v) \geq 0\}$$

which can be seen as a semigroup by taking the operation to be vector addition. In fact, it is finitely generated:

Proposition 2 (Gordan's Lemma)**.** *If σ is a cone, then S_σ is a finitely generated semigroup.*

Proof. Since σ is generated by vectors in N, we can take vectors $u_1, \dots, u_s$ in M that generate $\sigma^{\vee}$ as a cone. Take $K = \{\sum t_i u_i \mid 0 \leq t_i \leq 1\}$, which is compact as a subset of $\mathbb{R}^n$ in the Euclidean topology. Thus, since M is discrete, there are only finitely many points of M in K. Now, take $u \in S_\sigma$. We can write $u = \sum r_i u_i$ where $r_i = m_i + t_i$ for a non-negative integer m_i and $0 \leq t_i \leq 1$. Therefore $K \cap M$ generates S_σ and the claim follows. □

Next, we list a few useful relations of the semigroups generated by cones.

Proposition 3. *If τ is a face of σ in the fan Δ, then*

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$$

for some $u \in M$.

Proof. Since τ is a face of σ, τ is also a cone in Δ. In particular, $\tau = \sigma \cap u^{\perp}$ for some $u \in \sigma^{\vee} \cap M$. So, given $w \in S_\tau$ it is non-negative on τ as is u. However, w may be negative on some vectors of σ. However, $v \in \sigma$ is $v = r_1 v_1 + \cdots + r_t v_t$ where the v_i are the generators and the $r_i \in \mathbb{Q}$. So, $w(v) = \sum r_i w(v_i)$. Hence, there is a large enough positive integer $p > \max_i(|w(v_i)|)$ such that $w + p \cdot u$ is non-negative on all of σ. In other words, $w + p \cdot u \in S_\sigma$, so $w \in S_\sigma - p \cdot u$. Hence, $S_\tau \subseteq S_\sigma + \mathbb{Z}_{\geq 0}(-u)$. Conversely, if we take $w \in S_\sigma$ so that $w' = w + p(-u) \in S_\sigma + \mathbb{Z}_{\geq 0}(-u)$, we have that, for any $v \in \tau$, $w'(v) = w(v) + p(-u)(v) = w(v) \geq 0$. □

Proposition 4. *If $\sigma, \sigma' \in \Delta$ for some fan Δ and $\tau = \sigma \cap \sigma'$, then*

$$S_\tau = S_\sigma + S_{\sigma'}$$

Proof. Proposition 3 gives the inclusion $S_\sigma + S_{\sigma'} \subseteq S_\tau$. Also, by Proposition 1 we can take $u \in \sigma^{\vee} \cap (-\sigma')^{\vee} \cap M$ such that $\tau = \sigma \cap u^{\perp} = \sigma' \cap u^{\perp}$. Hence, $-u \in S_{\sigma'}$. Thus, since $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u)$, we must also have $S_\tau \subseteq S_\sigma + S_{\sigma'}$. □

4.2. **Gluing Affine Varieties.** If σ is a cone, then we define the affine ring corresponding to σ by $\mathbb{C}[S_\sigma]$. We denote by χ^u the element in the $\mathbb{C}$-algebra corresponding to the semigroup element u and require that $\chi^u\chi^{u'} = \chi^{u+u'}$ for $u + u' \in S_\sigma$. Each element of $\mathbb{C}[S_\sigma]$ is given by a finite sum $\sum c_i\chi^{u_i}$ for $c_i \in \mathbb{C}$ and $u_i \in S_\sigma$.

Now we can define the affine variety U_σ corresponding to a cone σ by setting

$$U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$$

which is the affine variety whose coordinate ring is $\mathbb{C}[S_\sigma]$. Here, $\operatorname{Spec}(R)$, even though it defines the set of prime ideals of R, can be thought of as the set of *maximal* ideals to get the intuition. In turn, the maximal ideals can be thought of as the points on $\mathbb{C}^n$.

Proposition 5. *If τ is a face of σ, then the map $U_\tau \to U_\sigma$ embeds U_τ as a principal open subset of U_σ.*

Proof. Taking $\tau = \sigma \cap u^\perp$ for $u \in S_\sigma$ and applying Proposition 3, we get that if $w' \in \tau$ then $w' = w - pu$. Hence, in $\mathbb{C}[S_\tau]$, we have $\chi^{w'} = \chi^{w-pu} = \chi^w / (\chi^u)^p$, i.e.

$$\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma][(\chi^u)^{-1}]$$

which proves the claim. □

This is all we needed to define the gluing. Consider a fan Δ. Since any two cones in $\sigma, \sigma' \in \Delta$ share a face, we have — by Proposition 5 — that there are injections $\varphi : U_{\sigma\cap\sigma'} \to U_\sigma$ and $\psi : U_{\sigma\cap\sigma'} \to U_{\sigma'}$. So, we make the identification

$$f : \varphi(U_{\sigma\cap\sigma'}) \to \psi(U_{\sigma\cap\sigma'})$$

by

$$x \mapsto \psi(\varphi^{-1}(x))$$

for all $x \in \varphi(U_{\sigma\cap\sigma'})$. The inverse is easily seen to be $y \mapsto \varphi(\psi^{-1}(y))$.

This can also be seen as given by the homomorphisms of $\mathbb{C}$-algebras — take the induced one from above, i.e. f^*. Alternatively, using Proposition 1, and the proof of Proposition 5, we can write

$$\mathbb{C}[S_{\sigma'}][(\chi^u)^{-1}] \cong \mathbb{C}[S_{\sigma'\cap\sigma}] \cong \mathbb{C}[S_\sigma][(\chi^u)^{-1}]$$

where $u \in (-\sigma')^\vee \cap \sigma^\vee$ and $\sigma' \cap \sigma = \sigma \cap u^\perp = \sigma' \cap u^\perp$. Hence, taking a homomorphism of semigroups $S_{\sigma'} \to S_\sigma$ that satisfies the above, gives a homomorphism of $\mathbb{C}$-algebras that defines the required gluing.

This patching of the U_σ for $\sigma \in \Delta$ gives the variety $X(\Delta)$.

Note that $\{0\} \subseteq \sigma$ is a face for any $\sigma \in \Delta$. Hence, $U_{\{0\}}$ "sits inside" U_σ. Going to the affine rings of both varieties, we see that this induces a homomorphism of $\mathbb{C}$-algebras $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_{\{0\}}]$. We turn to Proposition 4 for a justification. To see what $\mathbb{C}[S_{\{0\}}]$ really is, look at

$$S_{\{0\}} = \{u \in M \mid u(0) \geq 0\} = M$$

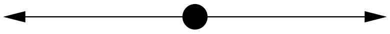

FIGURE 2. This fan is for $n = 1$. It has three cones: one generated by $-e_1$ ($\mathbb{R}_{\leq 0}$), another generated by e_1 ($\mathbb{R}_{\geq 0}$) and $\{0\}$.

where M is seen as a semigroup. If N is generated by $e_1, \ldots, e_n$ as a lattice, then M is generated by $\pm e_1^*, \ldots, \pm e_n^*$ as a semigroup. Putting, $X_i = \chi^{e_i^*}$ and $X_i^{-1} = \chi^{-e_i^*}$, we get that

$$\begin{aligned} U_{\{0\}} = \mathrm{Spec}(\mathbb{C}[S_{\{0\}}]) &= \mathrm{Spec}(\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]) \\ &\cong \mathrm{Spec}(\mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/(X_1Y_1 - 1, \cdots, X_nY_n - 1)) \\ &\cong (\mathbb{C}^*)^n = T_N \end{aligned}$$

So the torus is a principal open subset of all U_σ. Also, a natural action of T on $X(\Delta)$ exists which is an extension of the action of the torus on itself (i.e. for $a, b \in T$, we send (a, b) to $(ab)_i = a_i b_i$). For more details see [Fulton] and [Oda1]. This is why $X(\Delta)$ is called a *toric* variety.

5. EXAMPLES

Now we turn to a few descriptive examples. We shall look at $\mathbb{P}^1$, $\mathbb{P}^2$ and $\mathbb{P}^3$ as toric varieties and we will realize $\mathbb{P} \times \mathbb{C}$ as a toric variety.

Example 1. Consider the fan given by Figure 2. A direct consequence of the definition of S_σ is that $(\mathbb{R}_{\geq 0})^\vee$ is generated by $-e_1^*$ so S_σ is generated by X^{-1}. Similarly, $S_{\mathbb{R}_{\geq 0}}$ is generated by X. Finally, $S_{\{0\}}$ is given by both X and X^{-1}. (Note that both $\mathbb{C}[X]$ and $\mathbb{C}[X^{-1}]$ can be thought of as affine rings for $\mathbb{C}$ — they are isomorphic.) Thus, we make the identification $X \mapsto X^{-1}$. The claim is that these two copies of $\mathbb{C}$ glued in this way give $\mathbb{P}^1$. To see this, put $X = a_1/a_0$ (where a_0, a_1 are homogeneous coordinates in $\mathbb{P}^1$). Thus, $\mathbb{C}[X]$ becomes isomorphic to the open set V_0 of $\mathbb{P}^1$ for which $a_0 \neq 0$. Similarly, since $X^{-1} = a_0/a_1$, we have that $\mathbb{C}[X^{-1}]$ is isomorphic to the open set V_1 where $a_1 \neq 0$. Now, for $\mathbb{P}^1$, on $V_0 \cap V_1$ we require that both $a_0 \neq 0$ and $a_1 \neq 0$. This requirement is given in the toric variety by the isomorphism (on the intersection) $X \mapsto X^{-1}$ — a_1/a_0 is identified with a_0/a_1, i.e. $(1 : a_1/a_0) = (a_0/a_1 : 1)$.

Example 2. Now take the fan depicted on Figure 3. It consists of three cones and their faces. These are given by

$$\begin{aligned} \sigma_0 &= \langle e_1, e_2 \rangle \\ \sigma_1 &= \langle e_1, v_0 \rangle \\ \sigma_2 &= \langle v_0, e_2 \rangle \end{aligned}$$

To see what S_{σ_i} is, we need to look at $\sigma_i^\vee \cap M$. We know that $u \in M_\mathbb{R}$ is given by $u = ae_1^* + be_2^*$ since M is the dual lattice to N. If $v \in \sigma_0$, then $v = ce_1 + de_2$. For $u \in M_\mathbb{R}$ to

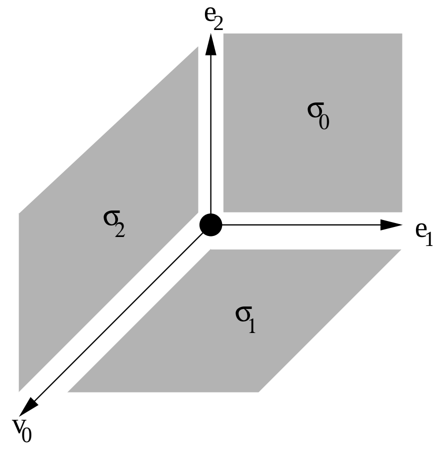

FIGURE 3. This fan (in the lattice $N = \mathbb{Z}^2$) generates a toric variety isomorphic to $\mathbb{P}^2$. The generating vectors for the cones are the basis vectors e_1 and e_2 and $v_0 = -e_1 - e_2$.

be in $\sigma_0^\vee \cap M$, we require that $u(v) \geq 0$, i.e.

$$\begin{aligned} u(v) &= (ae_1^* + be_2^*)(ce_1 + de_2) \\ &= (ae_1^*)(ce_1 + de_2) + (be_2^*)(ce_1 + de_2) \\ &= ac(e_1^*)(e_1) + bd(e_2^*)(e_2) \end{aligned}$$

and since $c \geq 0$ and $d \geq 0$ (see the section on cones), we must have $a \geq 0$ and $b \geq 0$ real numbers. Hence, e_1^* and e_2^* generate $\sigma_1^\vee$. Thus, putting $X = e_1^*$ and $Y = e_2^*$, we obtain that $\mathbb{C}[S_{\sigma_0}] = \mathbb{C}[X, Y]$.

Now, for $u \in M_\mathbb{R}$ to be in $\sigma_1^\vee \cap M$, we require that $u(v) \geq 0$. It is enough to ensure that $u(v_0) \geq 0$ and $u(e_1) \geq 0$ (on the generators), i.e.

$$\begin{aligned} u(v_0) &= (ae_1^* + be_2^*)(v_0) \\ &= (ae_1^*)(-e_1 - e_2) + (be_2^*)(-e_1 - e_2) \\ &= (ae_1^*)(-e_1) + (be_2^*)(-e_2) \geq 0 \\ u(e_1) &= (ae_1^* + be_2^*)(e_1) \\ &= (ae_1^*)(e_1) + (be_2^*)(e_1) \\ &= (ae_1^*)(e_1) \geq 0 \end{aligned}$$

From the second inequality $a \geq 0$. Substituting this in the first inequality, $b \leq 0$. In fact, u can be seen as a linear combination of $e_1^* - e_2^*$ and $-e_2^*$ since those are dual to the basis of the cone. We verify that this works (if $b' = -b$)

$$\begin{aligned} u(v) &= (a(e_1^* - e_2^*) + b'(-e_2^*))(ce_1 + dv_0) \\ &= a(e_1^* - e_2^*)(ce_1 + d(-e_1 - e_2)) + b'(-e_2^*))(ce_1 + d(-e_1 - e_2)) \\ &= ace_1^*(e_1) + \underbrace{ade_1^*(-de_1) + ad(-e_2^*)(-e_2)}_{=0} + b'd(-e_2^*)(-e_2) \end{aligned}$$

Hence, in the $\mathbb{C}$-algebra this corresponds to XY^{-1} and Y^{-1}. In other words, $\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[XY^{-1}, Y^{-1}]$. Exactly similarly, we obtain $\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[YX^{-1}, X^{-1}]$. Note that

$$\mathbb{C}[S_{\sigma_0}] \cong \mathbb{C}[S_{\sigma_1}] \cong \mathbb{C}[S_{\sigma_2}] \cong \mathbb{C}[x, y]$$

That is, each of the above cones defines an affine variety isomorphic to $\mathbb{C}^2$. In fact, taking $X = a_2/a_0$ and $Y = a_1/a_0$[3] we identify $\mathbb{C}[S_{\sigma_i}]$ to the open set V_i in $\mathbb{P}^2$. The identification glues two affine varieties along their shared face. For example, σ_0 and σ_1 share the face generated by only one vector — e_1. The dual cone of $\langle e_1 \rangle$ is given by X, Y, Y^{-1}. Therefore the gluing of U_{σ_0} and U_{σ_1} is realized on $U_{\langle e_1 \rangle}$ by sending $(X, Y) \mapsto (XY^{-1}, Y^{-1})$. Note that this translates to identifying

$$\left(1 : \frac{a_1}{a_0} : \frac{a_2}{a_0}\right) = \frac{a_0}{a_1}\left(1 : \frac{a_1}{a_0} : \frac{a_2}{a_0}\right) \tag{1}$$

which is what must hold for points in $V_0 \cap V_1$ back inside $\mathbb{P}^2$.

Similarly, $(X, Y) \mapsto (YX^{-1}, X^{-1})$ ensures compatibility of $V_0 \cap V_2$ and $XY^{-1}, Y^{-1}) \mapsto (YX^{-1}, X^{-1})$ takes care of $V_1 \cap V_2$. Finally, these same maps define the requirements on $V_0 \cap V_1 \cap V_2$. The verification of this is left to the reader (HINT: look at the maps as being of the form of Equation 1).

Example 3. To construct $\mathbb{P}^3$ out of four patches homeomorphic to $\mathbb{A}^3$, we need to impose correct gluing criteria. For this, we turn to the last example. If we can get four affine rings each isomorphic to $\mathbb{C}[x, y, z]$ and define appropriate gluing, we'll be set.

We start backwards. Take $X = a_1/a_0$, $Y = a_2/a_0$, $Z = a_3/a_0$ for the coordinates as in the previous two examples. As before, $\mathbb{C}[X, Y, Z]$ can be seen as $A(V_0)$ since we require $a_0 \neq 0$ with the assignments above. To get the corresponding affine ring for V_1, note that $X^{-1} = a_0/a_1$ requiring that $a_1 \geq 0$. Taking $\mathbb{C}[X^{-1}, YX^{-1}, ZX^{-1}]$ we obtain the required ring. The gluing $X, Y, Z \mapsto XY^{-1}, Y^{-1}, ZY^{-1}$ matches the requirement in $\mathbb{P}^3$ on $V_0 \cap V_1$. Similarly, we get the other two rings: $\mathbb{C}[XY^{-1}, Y^{-1}, ZY^{-1}]$ and $\mathbb{C}[XZ^{-1}, YZ^{-1}, Z^{-1}]$. It is easily verified that this setup gives the right construction (it is exactly similar to the previous example).

Also, one can generalize this idea to any dimension. In fact, the fan that gives this construction for dimension n can be taken to be the cones generated by any proper subset of

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[3] Again a_0, a_1, a_2 are homogeneous coordinates in $\mathbb{P}^2$.

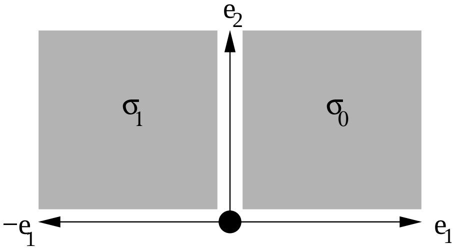

FIGURE 4. This fan is for $n = 2$. It has two main cones: one generated by e_1 and e_2 and another given by $-e_1$ and e_2. This fan gives $X(\Delta) = \mathbb{P} \times \mathbb{C}$.

$\{v_0, e_1, \ldots, e_n\}$ where the e_i are the standard basis vectors for the lattice and

$$v_0 = -e_1 - e_2 - \cdots - e_n.$$

It is a good exercise to verify this using the ideas from the second example.

Example 4. Consider the fan given by Figure 4. Following the ideas in the previous examples, one sees that $\mathbb{C}[S_{\sigma_0}] = \mathbb{C}[X, Y]$ and $\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[X^{-1}, Y]$. These two rings are isomorphic to $\mathbb{C}[x, y]$. Thus, U_{σ_0} and U_{σ_1} are both homeomorphic to $\mathbb{C}^2$. The gluing sends X, Y to X^{-1}, Y. Note that Y remains the same and the behavior of X is exactly as in the first example. In fact, $X(\Delta) \cong \mathbb{P} \times \mathbb{C}$.

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