MECH 289
Design Graphics

Single Axis Rotation for an Oriented Isometric (GCA2)

February 28, 2007

1 Introduction

For this exercise GCA2, to find the axis and angle of rotation necessary to bring a Cartesian frame into oriented isometric configuration, we begin with the simple trigonometry shown in Fig. 1.

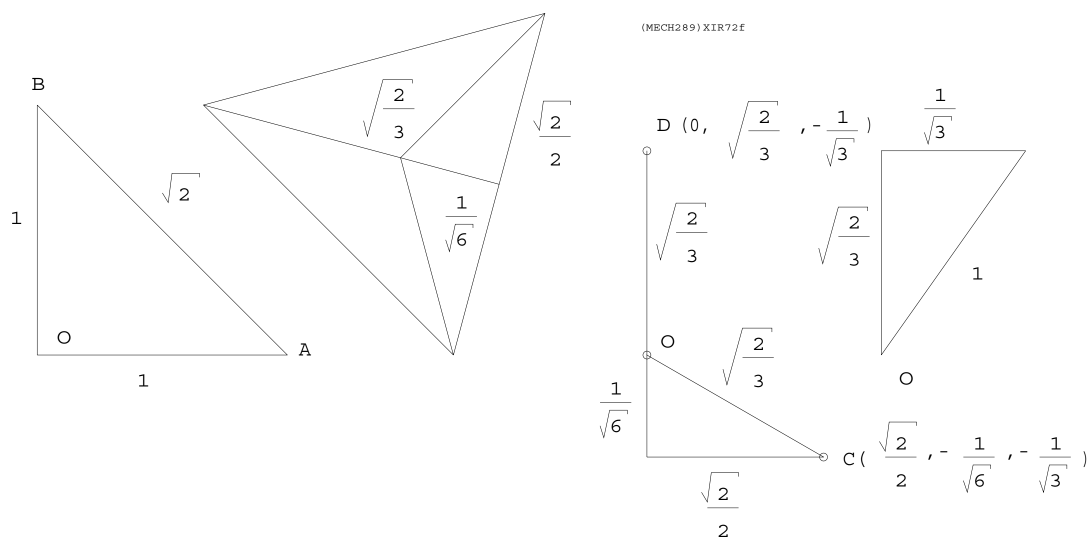

Figure 1: Rotational Displacement Trigonometry

On the left one sees x- and y- axes of unit length. Rotation is assumed to take place with origin $O(0,0,0)$ as the fixed point. The ends of the x- and y- axis segments are $A(a_1, a_2, a_3) = (1,0,0)$ and $B(b_1, b_2, b_3) = (0,1,0)$, respectively. The triangle with three spokes from centroid to vertices, immediately to the right, shows the length of the spokes. The three spokes, in the third image, are in relative orientation consistent with an isometric projection of three mutually orthogonal Cartesian axes of unit length, *i.e.*, the y-axis remains oriented as before. Axes appear foreshortened because we are looking down on a tetrahedral pyramid with an equilateral base with sides $\sqrt{2}$ long and three inclined edges of unit length. These are the three axes. In the two-view drawing on the right we see the desired isometric top view showing O fixed but A has moved to $C(c_1, c_2, c_3)$ and B to $D(d_1, d_2, d_3)$. Note the coordinates of C and D in Fig. 1.

2 Finding the Rotation Axis

Referring to Fig. 2, the rotation axis is on a line formed by the intersection of two planes

$$m\{M_0 : M_1 : M_2 : M_3\} \text{ and } n\{N_0 : N_1 : N_2 : N_3\}$$

These are the right bisectors of line segments $\vec{CA}$ and $\vec{DB}$. Since the rotation axis must be on O, which is always fixed, we can get a parametric line on O simply with a cross-product of the plane normals. Obviously

$$M_0 = 0, \quad M_1 = a_1 - c_1, \quad M_2 = a_2 - c_2, \quad M_3 = a_3 - c_3$$

$$N_0 = 0, \quad N_1 = b_1 - d_1, \quad N_2 = b_2 - d_2, \quad N_3 = b_3 - d_3$$

$$\mathbf{n} = \frac{\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} \times \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}}{\sqrt{M_1^2 + M_2^2 + M_3^2}\sqrt{N_1^2 + N_2^2 + N_3^2}}$$

so that the parametric equation of the axis line is $\mathbf{p} = \mathbf{n}t$ where t is a scalar free parameter, $\mathbf{n}$ is a unit vector in the direction of the line and $\mathbf{p}$ is the position vector of any point on the line.

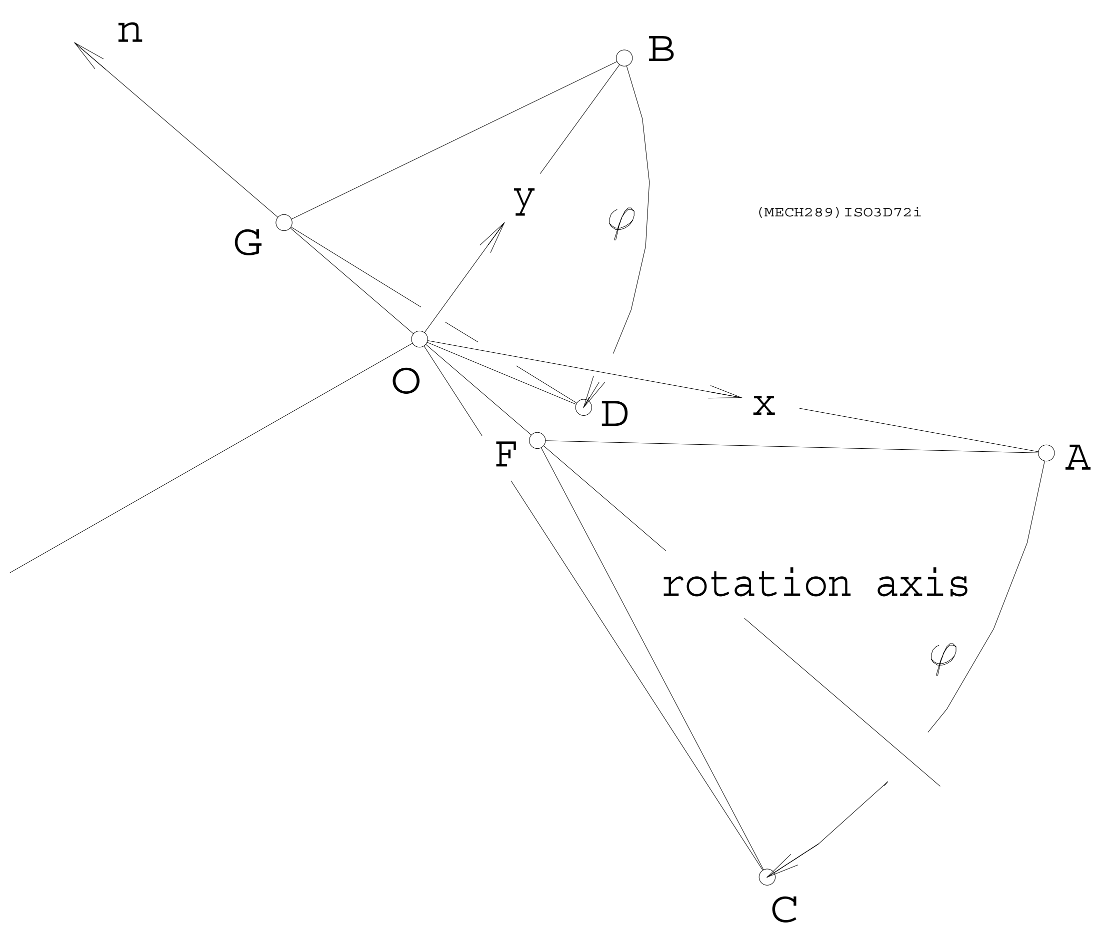

Figure 2: Rotational Displacement Trigonometry

3 Finding the Angle of Rotation

Now imagine planes normal to the axis line. One contains the segment $\overrightarrow{CA}$, the other, the segment $\overrightarrow{DB}$. Visualize two arms of the angle, in one plane, radiating from F on the axis to A and C, respectively. This angle is the same size as the one on another plane normal to the axis and with its vertex on the axis at G and its arms going to B and D, respectively. We need only consider one of these to get the rotation angle between either pair of arms. If you have trouble understanding this verbiage draw yourself a little free-hand sketch to clear things up. Let's deal with the triangle with vertices AFC. We have position vectors for everyone except F. Since the plane of this triangle is *not* on the origin –let's call the plane $f\{F_0 : F_1 : F_2 : F_3\}$, $F_0 \neq 0$– but we have $F_1 = n_1$, $F_2 = n_2$, $F_3 = n_3$ where

$$\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

it is merely necessary now to write the plane equation and observe that $A \in f$ to get F_0.

$$F_0 a_0 + F_1 a_1 + F_2 a_2 + F_3 a_3 = 0$$

Now to find the point F on the axis. This means finding t in the parametric equation. Since $\mathbf{n}$ is a position vector, $n_0 = 1$ and $f_i = n_i, \;\; i = 0, 1, 2, 3$.

$$F_0 f_0 + F_1 n_1 t + F_2 n_2 t + F_3 n_3 t = 0, \quad \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 \\ n_1 t \\ n_2 t \\ n_3 t \end{bmatrix}$$

The final step is to do some vector addition and subtraction and form the ratio that represents some trigonometric function of the desired angle ϕ. *E.g.*,

$$\sin\frac{\phi}{2} = \frac{|\mathbf{a} - \mathbf{c}|}{2|\mathbf{a} - \mathbf{f}|}$$

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