MECH 576
Geometry in Mechanics

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Equidistant Points from a Given Pair of Spheres

1 Introduction

Consider a given pair of spheres centred on $M(m_1, m_2, m_3)$ and $N(n_1, n_2, n_3)$ with respective radii m_4 and n_4. The following procedure to find the set of points equidistant from these two surfaces is suggested. It is merely a detailing and algorithmization of Stachel's elegant treatment [2] of confusion resolution in a global positioning system (GPS) problem where four spheres of electromagnetic wave front from earth satellites do not *quite* intersect so as to perfectly define the location as they would in the unlikely case where all four share an exact, common point of intersection. The idea is to select, as a good approximation of the position being sought, the centre of the sphere of least radius that is tangent to the four emitted by the satellites and received by the GPS locating instrument. Choosing three different pairs of spheres among the four given ones and finding the set of equidistant points from each pair then intersecting the three sets will yield the smallest tangent sphere among, as it will be seen, up to sixteen possible real tangent spheres. However, by applying a particular kind of simple, non-linear mapping called *Zyklographie* [1], the solution separates into two monomial equations of degree eight.

1.1 Critical Configurations

There are only two types of configuration where this procedure fails geometrically. Stachel points these out.

- If the the four spheres are generators of a Dupin's cyclide there is a one parameter set of solutions and
- If the Zyklographie produces a monomial with repeated roots.

Stachel also shows how this comes about when given sphere centres coincide with one or more of the equidistant surfaces that turn out to be hyperboloids of two sheets.

1.2 Planar Zyklographie

In 1988 Zsombor-Murray and Linder [3] presented a superficial introduction to *Zyklographie* by applying this non-linear mapping technique to solve the well known *Apollonius* problem of finding the (up to) eight circles tangent to three given ones. Here the space of *Laguerre* coordinates makes for a readily visible interpretation of this two dimensional problem by replacing the given constraint circles by points in a three dimensional space such that a point with three Cartesian coordinates represents a circle, *viz.*, $(x_1, x_2, x_3) \equiv (x, y, r)$ where (x, y) is a circle centre and r is its radius. All circles tangent to a given one, centre (m_1, m_2) and radius m_3, map to right cones (with apex angles of $\frac{\pi}{2}$) of revolution with all axes parallel to each other and normal to principal plane $x_3 = 0$. A typical constraint circle to which one desires to specify all possible tangent circles is given by an equation like Eq. 1.

$$(x_1 - m_1)^2 + (x_2 - m_2)^2 - m_3^2 = 0 \tag{1}$$

However the surface containing points that represent all possible circles tangent to the one represented by Eq. 1 is given by the quadric surface whose equation is Eq. 2.

$$(x_1 - m_1)^2 + (x_2 - m_2)^2 - (x_3 \pm m_3)^2 = 0 \tag{2}$$

If the $(+)$ sign is chosen then constraint circle and every circle tangent to it are in convex-to-convex contact. If the $(-)$ sign is chosen the set of tangent circles remain interior to the constraint circle until $x_3 = m_3$, an ambiguous case where the set and constraint circles *coincide*. When $x_3 > m_3$ the set of tangent circles engulf the constraint circle. These concepts and many sample cases are illustrated in [3].

2 Spatial Zyklographie

Much has been written on this subject. It is mostly German literature from the late nineteenth and early twentieth centuries, *e.g.*, Emil Müller's text on descriptive geometry [1]. Nevertheless, even without any deeper study, one may simply begin with two specimens of both Eq. 1 and 2 and raise these to three and four dimensions, respectively, before intersecting the two sets of tangent spheres to generate the surface of equidistant sphere centres. The two constraint spheres are

$$(x_1 - m_1)^2 + (x_2 - m_2)^2 + (x_3 - m_3)^2 - m_4^2 = 0 \quad (3)$$

$$(x_1 - n_1)^2 + (x_2 - n_2)^2 + (x_3 - n_3)^2 - n_4^2 = 0 \quad (4)$$

Replacing these descriptions by point sets in a four dimensional space, *i.e.*, given by *Laguerre* coordinates, yields Eqs. 5 and 6.

$$(x_1 - m_1)^2 + (x_2 - m_2)^2 + (x_3 - m_3)^2 - (x_4 - m_4)^2 = 0 \quad (5)$$

$$(x_1 - n_1)^2 + (x_2 - n_2)^2 + (x_3 - n_3)^2 - (x_4 - n_4)^2 = 0 \quad (6)$$

Subtracting Eq. 6 from Eq. 5 results in the linear equation of a four dimensional hyperplane, Eq. 7.

$$2[(n_1 - m_1)x_1 + (n_2 - m_2)x_2 + (n_3 - m_3)x_3 - (n_4 - m_4)x_4] + (m_1^2 + m_2^2 + m_3^2 - m_4^2) - (n_1^2 + n_2^2 + n_3^2 - n_4^2) = 0 \quad (7)$$

It is easy to see that eliminating x_4 between Eq. 7 and, say, Eq. 5 results in the quadric that is the locus of points equdistant from the spheres whose equations are Eq. 3 and 4. This quadric is represented by Eq. 8.

$$a_{00} + 2a_{01}x_1 + 2a_{02}x_2 + 2a_{03}x_3 + a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2 = 0 \quad (8)$$

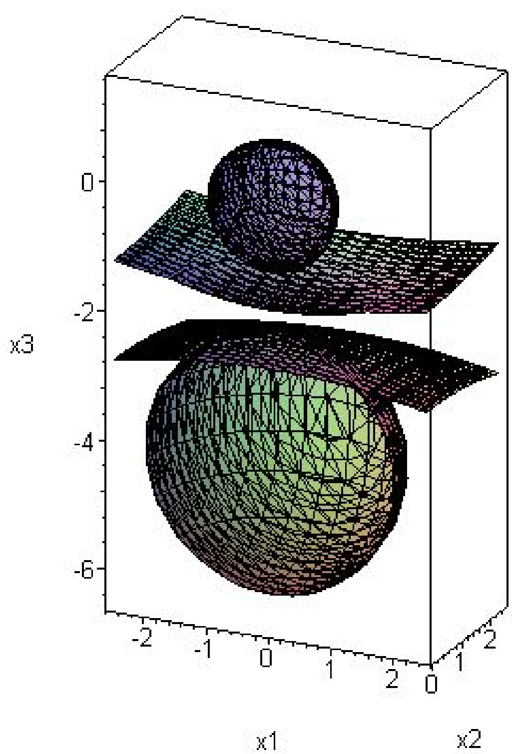

Figure 1: Two Spheres and the Equidistant Two-Sheet Hyperboloid of Revolution of Tangent Circle Centres

3 Example and Equation Coefficients

The example illustrated in Fig. 1 contains the two spheres where

$$m_1 = m_2 = m_3 = 0,\ \ m_4 = 1 \text{ and } n_1 = n_2 = 0,\ \ n_3 = -4,\ \ n_4 = 2$$

so as to yield the quadric, Eq. 9.

$$225 + 240x_3 - 4x_1^2 - 4x_2^2 + 60x_3^2 \tag{9}$$

Notice that everything where $x_2 < 0$ has been eliminated, showing the hollow interior of the spheres, to better illustrate that one branch of the hyperboloid indeed passes mid-way between the two spheres. The branch that intersects the larger sphere contains sphere centres of spheres that engulf both given tangent spheres. This is what happens when *both* m_4 and n_4 have the *same* sign. If m_4 and n_4 had opposite sign one would obtain spheres that alternately engulfed one given sphere and were external to the other. Coefficients a_{ij} in Eq. 8 are tabulated below as Eq. 10.

$$\begin{aligned} a_{00} = (g + k_4 m_4)k_0 - hk_4^2,\ \ g = k_0 + k_4 m_4,\ \ h = m_1^2 + m_2^2 + m_3^2 - m_4^2 \\ a_{0j} = gk_j + k_4^2 m_j,\ \ a_{ii} = k_i^2 - k_4^2,\ \ a_{ij} = k_i k_j,\ i = 1 \ldots 3,\ j = 1 \ldots 3 \\ k_0 = h - n_1^2 - n_2^2 - n_3^2 + n_4^2,\ \ k_i = n_i - m_i,\ \ k_4 = m_4 - n_4 \end{aligned} \tag{10}$$

The purpose of tabulating the ten quadric coefficients in detail is to help in implementing an algorithm. Any reduction in number of arithmetic operations involving many sums, differences, products and quotients, especially differences of similar magnitude, not only shortens computation but, more important, preserves precision.

References

[1] Krames, J.L. (1929) *Vorlesungen über darstellende Geometrie von Dr. Emil Müller, II Band, Die Zyklographie*, Franz Deuticke, Leipzig u. Wien.

[2] Stachel, H. (1996) "Why Shall We also Teach the Theory behind Engineering Graphics", *Technical Report No. 35*, TU-Wien, Institute for Geometry, 5pp.

[3] Zsombor-Murray, P.J. and Linder, K. (1988) "A Descriptive Geometric Approach to Circle Construction", *Proceedings of International Conference on Engineering Graphics and Descriptive Geometry*, TU-Wien, 88-07, v.2, pp.342-349.