1 A Conic and Two Tangent Lines

Examine Fig. 1. Shown there is a given conic $a$, represented by a general ellipse, and a given coplanar line $g\{G_0 : G_1 : G_2\}$. The problem is to find the point $X\{x_0 : x_1 : x_2\}$ on $a$ that is closest to $g$.

An equivalent problem, the one chosen to be solved, is to find tangent lines $x\{X_0 : X_1 : X_2\}$ parallel to $g$ and on $X$. Clearly there are two of these, the one shown and another, not shown, that is farthest from $g$, on the opposite side of $a$.

2 Constraint Equations

Since one seeks a solution in the Euclidean plane $x_0 = 1$ and the scalar equation, expressing $X \in a$, is Eq. 1.

$$
\begin{bmatrix}
1 & x_1 & x_2
\end{bmatrix}
\begin{bmatrix}
a_{00} & a_{01} & a_{02} \\
a_{01} & a_{11} & a_{12} \\
a_{02} & a_{12} & a_{22}
\end{bmatrix}
\begin{bmatrix}
1 \\
x_1 \\
x_2
\end{bmatrix}
= a_{00} + 2a_{01}x_1 + 2a_{02} + a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 = 0 \quad (1)
$$

The second constraint equation, Eq. 2, defines the coefficients or homogeneous coordinates of the polar line $x$ tangent to $a$ on $X$.

$$
\begin{bmatrix}
a_{00} & a_{01} & a_{02} \\
a_{01} & a_{11} & a_{12} \\
a_{02} & a_{12} & a_{22}
\end{bmatrix}
\begin{bmatrix}
1 \\
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
a_{00} + a_{01}x_1 + a_{02}x_2 \\
a_{01} + a_{11}x_1 + a_{12}x_2 \\
a_{02} + a_{12}x_1 + a_{22}x_2
\end{bmatrix}
= \begin{bmatrix}
X_0 \\
X_1 \\
X_2
\end{bmatrix}
= \lambda \begin{bmatrix}
\mu G_0 \\
G_1 \\
G_2
\end{bmatrix} \quad (2)
$$
The last vector states that $x$ and $g$ are parallel and the last two rows of Eq. 2 provide, together with Eq. 1, the three constraints necessary to handle the three variables $x_1$, $x_2$, $\lambda$. Rewriting these last two rows and eliminating $\lambda$ yields a second equation, Eq. 3, in only $x_1$ and $x_2$.

\[
(a_{01} + a_{11}x_1 + a_{12}x_2) - \lambda G_1 = 0 \\
(a_{02} + a_{12}x_1 + a_{22}x_2) - \lambda G_2 = 0 \\
(G_2a_{01} - G_1a_{02}) + (G_2a_{11} - G_1a_{12})x_1 + (G_2a_{12} - G_1a_{22})x_2 = 0
\]

(3)

3 A Univariate Quadratic

Eliminating $x_2$ from Eq. 1 with Eq. 3 produces the quadratic, Eq. 4.

\[
Ax_1^2 + Bx_1 + C = 0
\]

(4)

where

\[
A = (G_1^2a_{22} - 2G_1G_2a_{12} + G_2^2a_{11})(a_{11}a_{22} - a_{12}^2) \\
B = 2(G_1^2a_{22} - 2G_1G_2a_{12} + G_2^2a_{11})(a_{01}a_{22} - a_{02}a_{12}) \\
C = (2G_1G_2a_{12} - G_1^2a_{22})a_{02}^2 + G_2^2(a_{01}a_{22} - 2a_{02}a_{12})a_{01} + [G_1^2a_{22} + G_2(G_2a_{12} - 2G_1a_{22})a_{12}]a_{00}
\]

4 Normal Line on $X$

All that remains, after finding the two points $X$, is to find $G\{g_0 : g_1 : g_2\}$, via Eq. 6, the points of intersection of lines $y\{Y_0 : Y_1 : Y_2\}$, found with Eq. 5, and $g$. Line $y$ is on $X$ and normal to $g$. This is outlined below.

\[
Y_0 + Y_1x_1 + Y_2x_2 = 0, \quad Y_0 = -(Y_1x_1 + Y_2x_2), \quad Y_1 = -G_1, \quad Y_2 = G_2 \\
g_0 = G_1Y_2 - Y_1G_2, \quad g_1 = G_2Y_0 - G_0Y_2, \quad g_2 = G_0Y_1 - G_1Y_0
\]

(5)

(6)

The minimum and maximum distances $s$ to be compared are given by Eq. 7.

\[
s = \sqrt{(x_1 - g_1/g_0)^2 + (x_2 - g_2/g_0)^2}
\]

(7)

5 An Alternate Method Using Conic/Line Intersection

First the intersection points between $a$ and $g$ are found by eliminating, say, $x_2$ between the conic and line equations.
\[ a_{00} + 2a_{01}x_1 + 2a_{02}x_2 + a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = 0 \]
\[ G_0 + G_1x_1 + G_2x_2 = 0 \]
\[ (G_2^2a_{11} - 2G_1G_2a_{12} + G_1^2a_{22})x_1^2 \]
\[ + 2[G_0(G_1a_{22} - G_2a_{12}) + G_2(G_2a_{01} - G_1a_{02})]x_1 \]
\[ + (G_1^2a_{00} - 2G_0G_1a_{01} + G_0^2a_{11}) = 0 \]

Then the midpoint \( F \) between the two intersection points are found along with the conic centre point \( C \).

\[
F\{G_1(G_1a_{22} - 2G_2a_{12}) + G_2^2a_{11} : G_0(G_1a_{22} - G_2a_{12}) + G_2(G_2a_{01} - G_1a_{02})
: G_0(G_2a_{11} - G_1a_{12}) + G_1(G_1a_{02} - G_2a_{01})\}

C\{a_{11}a_{22} - a_{12}^2 : a_{01}a_{22} - a_{02}a_{12} : a_{01}a_{12} - a_{02}a_{11}\} \tag{8}

The line \( f\{F_0 : F_1 : F_2\} \) on \( F\{f_0 : f_1 : f_2\} \) and \( C\{c_0 : c_1 : c_2\} \) intersects the conic on the two points whereon the tangents to the conic are parallel to \( g \).

\[
f\{c_1f_2 - c_2f_1 : c_2f_0 - c_0f_2 : c_0f_1 - c_1f_0\}
\]

6 Numerical Example

Consider Fig. 2.

![Figure 2: Shortest and Farthest Distances Example](image)

The symmetric coefficient matrix of a standard form ellipse \(-4x_0^2 + x_1^2 + 4x_2^2 = 0\) has been rotated counterclockwise by an angle \( \tan^{-1} \frac{5}{12} \) and its centre translated to \((4, 3)\) by the transformation

\[
169 \left\{ \begin{bmatrix}
1 & -4 & -3 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{12}{13} & -\frac{5}{13} \\
0 & \frac{5}{13} & \frac{12}{13}
\end{bmatrix} \begin{bmatrix}
-4 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 4
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{12}{13} & \frac{5}{13} \\
0 & -\frac{12}{13} & \frac{13}{13}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
-4 & 1 & 0 \\
-3 & 0 & 1
\end{bmatrix} \right\}
\]
Notice premultiplications of the standard form coefficient matrix by cofactor (dual) matrices of point vector rotation (inner to effect pre-rotation) and point vector translation (outer to effect post-translation). Postmultiplications are in the same inner/outer sequence and these are the transposes of their premultiplier counterparts. The scalar multiplier $13^2 = 169$ renders the resulting matrix, that produces the ellipse in a general disposition, free of fractional elements.

$$4317x_0^2 - 872x_0x_1 - 2166x_0x_2 + 244x_1^2 - 360x_1x_2 + 601x_2^2 = 0$$

Intersecting the given line $x_2 = 0$ with the general ellipse gives the complex conjugate point pair \{1 : $\frac{109}{61} \pm \frac{13}{122}\sqrt{1277i}$ : 0\}. The midpoint between these two is real however; \{1 : $\frac{109}{61}$ : 0\}. Joining it to the conic centre point \{1 : 4 : 3\} produces the line

$$-327x_0 + 183x_1 + 353x_2 = 0$$

that intersects the conic on points that are, respectively, the closest to and farthest from the given line. Alternately, putting the three line coordinates $G_0 = -327$, $G_1 = 183$, $G_2 = 353$ in Eq. 8 and solving for two values of $x_1$ yields the vertical lines $x_1 - 4.88641 = 0$ and $x_1 - 3.11359$. These establish the location of the same two points on the conic.

### 7 Distance via Discriminant

Recall the quadratic univariate, in terms of $x_1$, that results from the intersection of the given line and quadric.

$$\begin{align*}
(G_0^2 a_{11} - 2G_1 G_2 a_{12} + G_2^2 a_{22})x_1^2 \\
+ 2[G_0(G_1 a_{12} - G_2 a_{11}) + G_2(G_2 a_{01} - G_1 a_{02})]x_1 \\
+ (G_2^2 a_{00} - 2G_0 G_2 a_{02} + G_2^2 a_{22}) &= 0
\end{align*}$$

Taking the derivative of this univariate with respect to $x_1$ yields a second equation. Eliminating $x_1$, removing factor

$$2[2G_2^2(G_0^2 a_{11} - 2G_1 G_2 a_{12} + G_2^2 a_{22})]$$

devoid of $G_0$ produces an expression for $G_0$.

$$\begin{align*}
(a_{11} a_{22} - a_{12}^2)G_0^2 + 2[(G_2 a_{12} - G_1 a_{22})a_{01} + 2[(G_1 a_{12} - G_2 a_{11})a_{02}]G_0 \\
+ (G_2^2 a_{11} - 2G_1 G_2 a_{12} + G_2^2 a_{22})a_{00} + G_2(2G_1 a_{02} - G_2 a_{01})a_{01} - G_1^2 a_{02} &= 0
\end{align*}$$

Assuming that one obtains two real roots $G_0'$ and $G_0''$ one may configure three versions of the line equation.

$$G_1 x_1 + G_2 x_2 + \begin{pmatrix} G_0' \\ G_0'' \end{pmatrix} = 0$$
Normalizing on the line normal direction numbers, \textit{i.e.}, dividing these three equations by $\sqrt{G_1^2 + G_2^2}$ leaves the differences, say,

$$|G_0 - G'_0| > |G_0 - G''_0|$$

so that one obtains the distances to the closer and farther tangent lines to the conic, parallel to the original line, the equation that contains $G_0$. 

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