MECH 576
Computer Graphics and Geometric Modeling
Shortest Distance from Line to Coplanar Conic

April 20, 2007

1 A Conic and Two Tangent Lines

Examine Fig. 1. Shown there is a given conic a, represented by a general ellipse, and a given coplanar line $g\{G_0 : G_1 : G_2\}$. The problem is to find the point $X\{x_0 : x_1 : x_2\}$ on a that is closest to g.

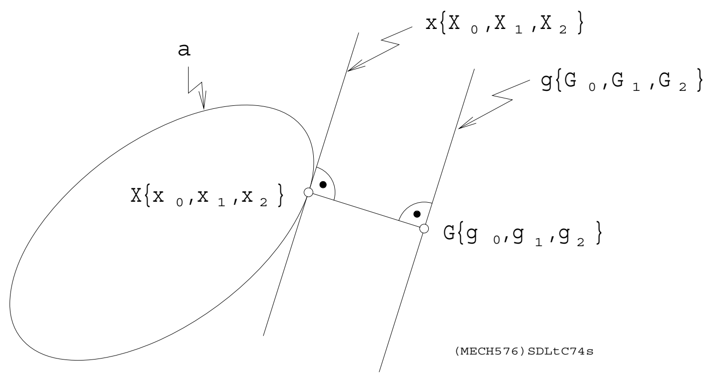

Figure 1: Line and Conic

An equivalent problem, the one chosen to be solved, is to find tangent lines $x\{X_0 : X_1 : X_2\}$ parallel to g and on X. Clearly there are two of these, the one shown and another, not shown, that is *farthest* from g, on the opposite side of a.

2 Constraint Equations

Since one seeks a solution in the Euclidean plane $x_0 = 1$ and the scalar equation, expressing $X \in a$, is Eq. 1.

$$[1\ x_1\ x_2]\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} = a_{00} + 2a_{01}x_1 + 2a_{02} + a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 = 0 \qquad (1)$$

The second constraint equation, Eq. 2, defines the coefficients or homogeneous coordinates of the polar line x tangent to a on X.

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{00} + a_{01}x_1 + a_{02}x_2 \\ a_{01} + a_{11}x_1 + a_{12}x_2 \\ a_{02} + a_{12}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = \lambda \begin{bmatrix} \mu G_0 \\ G_1 \\ G_2 \end{bmatrix} \qquad (2)$$

The last vector states that x and g are parallel and the last two rows of Eq. 2 provide, together with Eq. 1, the three constraints necessary to handle the three variables x_1, x_2, λ. Rewriting these last two rows and eliminating λ yields a second equation, Eq. 3, in only x_1 and x_2.

$$\begin{aligned}(a_{01} + a_{11}x_1 + a_{12}x_2) - \lambda G_1 &= 0 \\ (a_{02} + a_{12}x_1 + a_{22}x_2) - \lambda G_2 &= 0 \\ (G_2a_{01} - G_1a_{02}) + (G_2a_{11} - G_1a_{12})x_1 + (G_2a_{12} - G_1a_{22})x_2 &= 0 \end{aligned} \tag{3}$$

3 A Univariate Quadratic

Eliminating x_2 from Eq. 1 with Eq. 3 produces the quadratic, Eq. 4.

$$Ax_1^2 + Bx_1 + C = 0 \tag{4}$$

where

$$\begin{aligned} A &= (G_1^2a_{22} - 2G_1G_2a_{12} + G_2^2a_{11})(a_{11}a_{22} - a_{12}^2) \\ B &= 2(G_1^2a_{22} - 2G_1G_2a_{12} + G_2^2a_{11})(a_{01}a_{22} - a_{02}a_{12}) \\ C &= (2G_1G_2a_{12} - G_1^2a_{22}^2)a_{02}^2 + G_2^2(a_{01}a_{22} - 2a_{02}a_{12})a_{01} + [G_1^2a_{22}^2 + G_2(G_2a_{12} - 2G_1a_{22})a_{12}]a_{00} \end{aligned}$$

4 Normal Line on X

All that remains, after finding the two points X, is to find $G\{g_0 : g_1 : g_2\}$, via Eq. 6, the points of intersection of lines $y\{Y_0 : Y_1 : Y_2\}$, found with Eq. 5, and g. Line y is on X and normal to g. This is outlined below.

$$Y_0 + Y_1x_1 + Y_2x_2 = 0, \;\; Y_0 = -(Y_1x_1 + Y_2x_2), \;\; Y_1 = -G_1, \;\; Y_2 = G_2 \tag{5}$$

$$g_0 = G_1Y_2 - Y_1G_2, \;\; g_1 = G_2Y_0 - G_0Y_2, \;\; g_2 = G_0Y_1 - G_1Y_0 \tag{6}$$

The minimum and maximum distances s to be compared are given by Eq. 7.

$$s = \sqrt{(x_1 - g_1/g_0)^2 + (x_2 - g_2/g_0)^2} \tag{7}$$

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MECH 576
Computer Graphics and Geometric Modeling
Shortest Distance from Point to Coplanar Conic

April 24, 2007

1 A Conic and Four Tangent Circles

Examine Fig. 1. Shown there is a given conic a, represented by a general ellipse, and a given coplanar point $P\{p_0 : p_1 : p_2\}$. The problem is to find the point $X\{x_0 : x_1 : x_2\}$ on a that is closest to P.

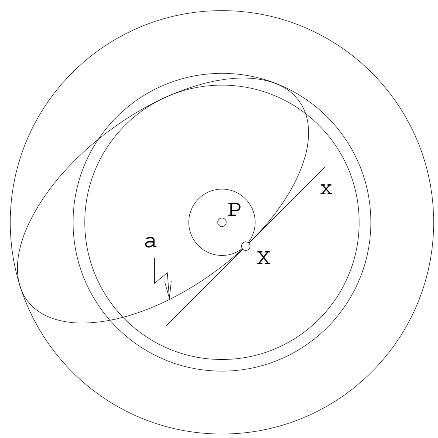

Figure 1: Point and Conic

An equivalent problem, the one chosen to be solved, is to find lines $x\{X_0 : X_1 : X_2\}$ that are cotangential to a and a circle centred on P and on X the point of tangency. Clearly there are four of these. The one shown on X near P is the closest. Another, not shown, is *farthest* from P. It is associated with the tangent circle of greatest radius and touches on the opposite side of a. There are two more tangent circles, concentric on P, shown in this example indicating that this problem configuration admits four real solutions

2 Constraint Equations

One seeks a solution in the Euclidean plane with $x_0 = 1$. There is no loss in generality if P is on the origin and the scalar equation, expressing $X \in k$, where k is the circle centred on P, is Eq. 1.

$$[1\ x_1\ x_2] \begin{bmatrix} -r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 - r^2 = 0 \tag{1}$$

The second constraint equation, Eq. 2, defines the coefficients or homogeneous coordinates of the polar line x tangent to a on X.

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{00} + a_{01}x_1 + a_{02}x_2 \\ a_{01} + a_{11}x_1 + a_{12}x_2 \\ a_{02} + a_{12}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = \lambda \begin{bmatrix} r^2 \\ x_1 \\ x_2 \end{bmatrix} \tag{2}$$

The last vector states that x is tangent on k and the three rows of Eq. 2 provide, together with Eq. 1, the four constraints necessary to handle the four variables $x_1,\ x_2,\ r^2,\ \lambda$. First λ is eliminated among the equations expressed by the three rows of Eq. 2. Then r^2 is eliminated among these two resultants and Eq. 1 to yield the two cubics Eq. 3.

$$\begin{aligned} a_{00}x_0^2x_1 + 2a_{01}x_0x_1^2 + a_{02}x_0x_1x_2 + a_{01}x_0x_2^2 + a_{11}x_1^3 + a_{12}x_1^2x_2 + a_{12}x_1x_2^2 + a_{12}x_2^3 = 0 \\ a_{00}x_0^2x_2 + a_{02}x_0x_1^2 + a_{01}x_0x_1x_2 + 2a_{02}x_0x_2^2 + a_{12}x_1^3 + a_{22}x_1^2x_2 + a_{12}x_1x_2^2 + a_{22}x_2^3 = 0 \end{aligned} \tag{3}$$

The variable $x_0 = 1$ is included to present homogeneous cubics with terms ordered lexographically.

3 A Univariate Quartic

The resultant of Eq. 3, Eq 4, is a univariate polynomial of degree six in x_1, a product of linear, quadratic and quartic factors. The linear and quadratic factors represent degenerate solutions. The geometric significance of these have not been determined but it is clear they cannot represent complete, valid solutions because the linear factor is devoid of any conic coefficients while the quadratic factor lacks $a_{11},\ a_{12}$ and a_{22}.

$$x_1(Ax_1^2 + Bx_1 + C)(Dx_1^4 + Ex_1^3 + Fx_1^2 + Gx_1 + H) = 0 \tag{4}$$

4 The Coefficients

The eight coefficients in Eq. 4 are tabulated below in Eq. 5 to show the computational effort required to compute them. When solving such a problem one would evaluate the quartic. The coefficients A, B and C indicate that the quadratic can generate only complex values of x_1.

$$\begin{aligned} A = a_{01}^2 + a_{02}^2, \quad B = a_{00}a_{01}, \quad C = a_{00}^2 \\ D = (a_{11}a_{22} - a_{12}^2)(a_{11}^2 + a_{22}^2 + 4a_{12}^2 - 2a_{11}a_{22}) \\ E = 2\left\{\left[(4a_{12}a_{22} - a_{11}a_{12})a_{11} - (6a_{12}^2 + a_{22}^2)\right] a_{02}\right. \\ \left. + \left[(5a_{12}^2 + a_{22}^2)a_{22} + (2a_{11}a_{22} - a_{12}^2 - 3a_{22}^2)a_{11}\right] a_{01}\right\} \\ F = (2a_{11}a_{22} - 12a_{12}^2 - a_{22}^2)a_{02}^2 + [6(2a_{22} - a_{11})a_{02}a_{12} + (5a_{11} - 4a_{22})a_{01}a_{22}]a_{01} \\ +[(4a_{12}^2 + a_{22}^2)a_{22} + (a_{11} - a_{22})a_{11}a_{22}]a_{00} \\ G = 2\left\{[a_{02}^2a_{22} + (a_{01}a_{12} - 2a_{02}a_{12})a_{01}]a_{01} - 2a_{02}^3a_{12}\right. \\ \left. [(2a_{22} - a_{11})a_{02}a12 + (a_{11}a_{22} + a_{12}^2 - a_{22}^2)a_{01}]a_{00}\right\} \\ H = (a_{01}^2a_{22} - 2a_{01}a_{02}a_{12} + a_{00}a_{12}^2)a_{00} \end{aligned} \tag{5}$$

5 An Exercise

Redo this problem with $a_{01} = a_{02} = a_{12} = 0$ and $P \neq P(0,0)$. Does this lead to a more compact solution? Indeed it does. Using the standard form conic scalar equation coefficient matrix and that of a circle centred on $P(p_1, p_2)$, for the common polar line relation, and the scalar equation for points on a circle,

$$\begin{bmatrix} a_{00} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{bmatrix}, \quad \begin{bmatrix} p_1^2 + p_2^2 - r^2 & -p_1 & -p_2 \\ -p_1 & 1 & 0 \\ -p_2 & 0 & 1 \end{bmatrix}, \quad (x_1 - p_1)^2 + (x_2 - p_2)^2 - r^2 = 0$$

yields the following quartic univariate in x_1 after removing a degenerate factor $(x_1 - p_1)$.

$$Px_1^4 + Qx_1^3 + Rx_1^2 + Sx_1 + S = 0$$

where

$$\begin{aligned} P = a_{11}(a_{11} - a_{22})^2, \quad Q = 2a_{11}a_{22}p_1(a_{11} - a_{22}) \\ R = a_{11}a_{22}(a_{11}p_2^2 - 2a_{00} + a_{22}p_1^2) + a_{00}(a_{11}^2 + a_{22}^2) \\ S = 2a_{00}a_{22}p_1(a_{11} - a_{22}), \quad T = a_{00}a_{22}^2p_1^2 \end{aligned}$$

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MECH 576
Computer Graphics and Geometric Modeling
Shortest Distance from Line to Quadric

May 8, 2007

1 Quadric, Tangent Line and Tangent Plane

Examine Fig. 1. Shown there is a given quadric a, represented by a general hyperboloid of one sheet, and a given line $\mathcal{G}_r\{1:0:0:0:0:0\}$. There is no loss in generality if the x_1-axis is chosen to be the given line. The problem is to find the point $X\{x_0 : x_1 : x_2 : x_3\}$ on a that is closest to $\mathcal{G}$.

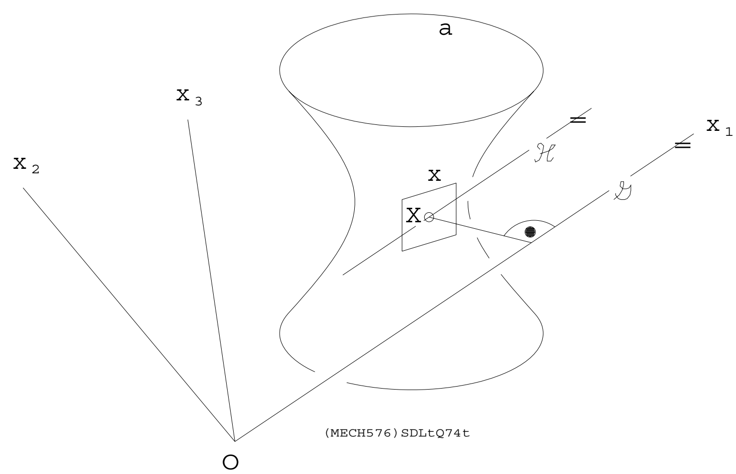

Figure 1: Line and Quadric

This is a two stage problem. The first stage involves reducing this spatial problem into a planar one involving extreme distances from point to conic. Then one may invoke the four concentric tangent circles, centred on the point, to the conic model that admits a quartic solution.

2 Constraint Equations

In addition to x the tangent plane to and the point X on a, a line $\mathcal{H}$ parallel to $\mathcal{G}$ is introduced. It is observed that

$$\mathcal{H} \parallel \mathcal{G},\ \ \mathcal{H} \in x,\ \ X \in \mathcal{H},\ \ X \in a$$

so one may immediately write the Plücker coordinates of $\mathcal{H}$ and polar tangency relation for x with respect to a, *i.e.*, $X \in x$. Solutions in Euclidean space allow setting $x_0 = 1$.

$$\begin{array}{c}\mathcal{H}_r\{1:0:0:0:h_{31}:h_{12}\} \equiv \mathcal{H}_a\{0:H_{02}:H_{03}:1:0:0\} \\ x \equiv \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} a_{00}+a_{01}x_1+a_{02}x_2+a_{03}x_3 \\ a_{01}+a_{11}x_1+a_{12}x_2+a_{13}x_3 \\ a_{02}+a_{12}x_1+a_{22}x_2+a_{23}x_3 \\ a_{03}+a_{13}x_1+a_{23}x_2+a_{33}x_3 \end{bmatrix}\end{array} \tag{1}$$

Now the conditions $\mathcal{H} \in x$ and $X \in \mathcal{H}$ are expressed with the conventional relations that declare, respectively, the nonexistence of the intersection of line and plane to denote containment and the nonexistence of a plane spanned by line and point to denote point-on-line. This will relate X_i to h_{ij} in the first instance and x_i to H_{ij} in the second.

$$\begin{array}{ll} h_{01}X_1 + h_{02}X_2 + h_{03}X_3 = 0 & \rightarrow X_1 = 0 \\ -h_{01}X_0 + h_{12}X_2 - h_{31}X_3 = 0 & \rightarrow X_0 - h_{12}X_2 + h_{31}X_3 = 0 \\ -h_{02}X_0 - h_{12}X_1 + h_{23}X_3 = 0 & \\ -h_{03}X_0 + h_{31}X_1 - h_{23}X_2 = 0 & \end{array} \tag{2}$$

$$\begin{array}{ll} H_{01}x_1 + H_{02}x_2 + H_{03}x_3 = 0 & \\ -H_{01}x_0 + H_{12}x_2 - H_{31}x_3 = 0 & \\ -H_{02}x_0 - H_{12}x_1 + H_{23}x_3 = 0 & \rightarrow x_3 - h_{31} = 0 \\ -H_{03}x_0 + H_{31}x_1 - H_{23}x_2 = 0 & \rightarrow x_2 + h_{12} = 0 \end{array} \tag{3}$$

Combining the first result from Eq. 2 with the second row of Eq. 1 produces a constraint, Eq. 4, on x_i in terms of conic coefficients a_{ij}.

$$a_{01} + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \tag{4}$$

Then both results from Eq. 3 are used to eliminate h_{ij} from the second result in Eq. 2 to produce $X_0 - X_2x_2 + X_3x_3 = 0$ which can be reformulated in terms of quadric coefficients a_{ij} and point variables x_i by using the remaining three lines in Eq. 1. The second constraint is Eq. 5.

$$(a_{00}+a_{01}x_1+a_{02}x_2+a_{03}x_3)+(a_{02}+a_{12}x_1+a_{22}x_2+a_{23}x_3)x_2+(a_{02}+a_{12}x_1+a_{23}x_2+a_{33}x_3)x_3 = 0 \tag{5}$$

Eliminating x_1 between Eq. 4 and Eq. 5 gives a conic in x_2 and x_3, *i.e.*, Eq. 6.

$$\begin{array}{c}(a_{00}a_{11} - a_{01}^2) + 2(a_{02}a_{11} - a_{01}a_{12})x_2 + 2(a_{03}a_{11} - a_{01}a_{13})x_3 \\ (a_{11}a_{22} - a_{12}^2)x_2^2 + 2(a_{11}a_{23} - a_{12}a_{13})x_2x_3 + (a_{11}a_{33} - a_{13}^2)x_3^2 = 0\end{array} \tag{6}$$

This conic, really a cylinder parallel to axis $x_1 = 0$, replaces a in the exercise where the extreme distances from a point to a conic were derived, thus beginning stage two. If the quadric point equation

$$a_{00} + 2a_{01}x_1 + 2a_{02}x_2 + 2a_{03}x_3 + a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2 = 0$$

is used to eliminate x_1, instead of Eq. 5, the resulting bivariate is identical to Eq. 6. One cannot obtain a univariate at this stage. Only by eliminating x_1, however, will a point projection of lines $\mathcal{G}$ and $\mathcal{H}$ appear together with the conic Eq. 6.

MECH 576
Computer Graphics and Geometric Modeling
Shortest Distance from Point to Quadric

May 8, 2007

1 A Quadric and Six Tangent Spheres

Examine Fig. 1. Shown there is a given quadric a, represented by a general standard form hyperboloid of one sheet, and a given arbitrary point $P\{p_0 : p_1 : p_2\}$. The problem is to find the point $X\{x_0 : x_1 : x_2\}$ on a that is closest to P.

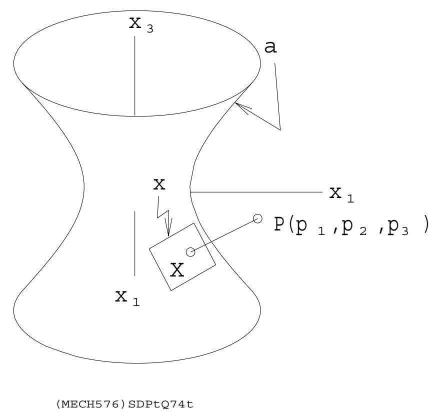

Figure 1: Point and Quadric

It may be difficult to visualize six real spheres concentric on P and tangent to a. As an exercise in geometric thinking imagine first an ellipsoid whose principal axis lengths are all different then morph this in your mind's eye to a rectangular box. Now place the point inside so that it is at different distances from all six sides. Then it would be easy to find the six distinct concentric tangent spheres centred on P, each touching a face. Thus one may be led to expect a univariate polynomial solution of degree six but one should not be too certain because such intuition is sometimes a useful but capricious guide.

2 Constraint Equations

Consider the symmetric coefficient matrices for the standard form quadric and a sphere centred on P and having radius r and the equation of that sphere, written as Eq. 1, below thr two matrces.

$$\begin{bmatrix} a_{00} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{bmatrix}, \begin{bmatrix} p_1^2+p_2^2+p_3^2-r^2 & -p_1 & -p_2 & -p_3 \\ -p_1 & 1 & 0 & 0 \\ -p_2 & 0 & 1 & 0 \\ -p_3 & 0 & 0 & 1 \end{bmatrix}$$

$$(x_1-p_1)^2+(x_2-p_2)^2+(x_3-p_3)^2-r^2=0 \quad (1)$$

Together with Eq. 1 the relation, obtained by equating common points X on common polar planes x, produces the five equations necessary to evaluate variables $r^2,\ \lambda,\ x_1,\ x_2,\ x_3$.

$$a_{00} - \lambda(p_1^2 + p_2^2 + p_3^2 - r^2 - p_1x_1 - p_2x_2 - p_2x_3) = 0 \tag{2}$$
$$a_{11}x_1 - \lambda(x_1 - p_1) = 0 \tag{3}$$
$$a_{22}x_2 - \lambda(x_2 - p_2) = 0 \tag{4}$$
$$a_{33}x_3 - \lambda(x_3 - p_3) = 0 \tag{5}$$

First r^2 is eliminated between Eq. 1 and Eq. 2 to produce a fourth equation Eq. 6 that contains the three point coordinates and the multiplier λ.

$$a_{00} - \lambda(p_1x_1 + p_2x_2 + p_3x_3 - x_1^2 + x_2^2 + x_3^2) = 0 \tag{6}$$

Elimination sequence is important to avoid trouble. Three equations in the three point coordinates are produced by using Eq. 3 to eliminate λ from Eq. 4, Eq. 5 and Eq. 6. Two bivariates in products x_1x_2 and x_1x_3, respectively, and a conic in x_3 is the result. Since x_2 does not appear in the second of these intermediate equations, x_2 is chosen for elimination between the first and third to produce a second equation devoid of x_2. Finally, x_3 is eliminated from the two bivariates now available in x_1 and x_3. This is a univariate in x_1 of degree seven that contains a degenerate linear factor as shown in Eq. 7.

$$(x_1 - p_1)(Px_1^6 + Qx_1^5 + Rx_1^4 + Sx_1^3 + Tx_1^2 + Ux_1 + V = 0 \tag{7}$$

The seven coefficients of the univariate are written below.

$$P = a_{11}(a_{11} - a_{22})^2(a_{11} - a_{33})^2$$
$$Q := 2a_{11}(a_{11} - a_{22})(a_{11} - a_{33})(a_{11}a_{22} + a_{11}a_{33} - 2a_{22}a_{33})p_1^2$$
$$R = a_{11}\left[6a_{22}^2a_{33}^2p_1^2 + a_{11}\left\{a_{22}a_{33}[a_{33}(p_2^2 - 6p_1^2) + a_{22}(p_3^2 - 6p_1^2)]\right.\right.$$
$$\left.\left.+a_{11}\left(a_{33}^2p_1^2 + a_{22}[a_{22}p_1^2 + 2a_{33}(2p_1^2 - p_2^2 - p_3^2)] + a_{11}(a_{22}p_2^2 + a_{33}p_3^2)\right)\right\}\right]$$
$$+a_{00}\{a_{22}^2a_{33}^2 + a_{11}\left(a_{11}[a_{11}(a_{11} - 2a_{22} - 2a_{33}) + a_{22}(a_{22} - 4a_{33}) + a_{33}^2] - 2a_{22}a_{33}(a_{22} + a_{33})\right)\}$$
$$S = 2p_1\left[a_{11}a_{33}\{a_{11}\left(a_{11}a_{22}(p_2^2 + p_3^2) + a_{22}[a_{22}(p_1^2 - p_3^2) + a_{33}(p_1^2 - p_2^2)]\right) - 2a_{22}^2a_{33}\}\right.$$
$$\left.+a_{00}\{a_{11}\left(a_{11}[a_{11}(a_{22} + a_{33}) - a_{22}(a_{22} - 4a_{33}) - a_{33}^2] + 3a_{22}a_{33}(a_{22} + a_{33})\right) - 2a_{22}^2a_{33}^2\}\right]$$
$$T = \left\{a_{11}a_{22}[a_{22}a_{33}^2p_1^2 + a_{11}a_{33}(a_{33}p_2^2 + a_{22}p_3^2)]\right.$$
$$\left.+\ a_{00}\{6a_{22}a_{33}^2 + a_{11}\left(a_{11}[a_{22}(a_{22} + 4a_{33}) + a_{33}^2] - 6a_{22}a_{33}(a_{22} + a_{33})\right)\}\right\}p_1^2$$
$$U = 2a_{00}a_{22}a_{33}(a_{11}a_{22} + a_{11}a_{33} - 2a_{22}a_{33})$$
$$V = a_{00}a_{22}^2a_{33}^2p_1^4$$

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MECH 576
Computer Graphics and Geometric Modelling

April 10, 2007

Gram-Schmidt Orthogonalization and Upper Triangularization

1 Orthogonalization

It is shown how a basis, represented by three column vectors in the 3×3 matrix $\mathbf{A}$, is converted to the three orthogonal vectors in $\mathbf{A}^*$ that span the same three dimensional homogeneous vector space. The first column $\mathbf{a}^*_{1j}$ is taken as the first principal Cartesian axis direction numbers.

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \rightarrow \mathbf{A}^* = \begin{bmatrix} a^*_{00} & a^*_{01} & a^*_{02} \\ a^*_{10} & a^*_{11} & a^*_{12} \\ a^*_{20} & a^*_{21} & a^*_{22} \end{bmatrix}$$

The procedure is outlined below.

- Choose $\mathbf{a}^*_{i0} = \mathbf{a}_{i0}$.
- Project $\mathbf{a}_{i1}$ on $\mathbf{a}^*_{i0}$ to obtain the component $\mathbf{a}_{i1\|i0} = \frac{(\mathbf{a}_{i1} \cdot \mathbf{a}_{i0})}{\mathbf{a}^2_{i0}} \mathbf{a}_{i0}$.
- Subtract to obtain $\mathbf{a}^*_{i1} = \mathbf{a}_{i1\perp i0} = \mathbf{a}_{i1} - \mathbf{a}_{i1\|i0}$.
- Project $\mathbf{a}_{i2}$ on $\mathbf{a}^*_{i0}$ to obtain the component $\mathbf{a}_{i2\|i0} = \frac{(\mathbf{a}_{i2} \cdot \mathbf{a}_{i0})}{\mathbf{a}^2_{i0}} \mathbf{a}_{i0}$.
- Subtract to obtain $\mathbf{a}'_{i2} = \mathbf{a}_{i2\perp i0} = \mathbf{a}_{i2} - \mathbf{a}_{i2\|i0}$.
- Project $\mathbf{a}'_{i2}$ on $\mathbf{a}^*_{i1}$ to obtain the component $\mathbf{a}_{i2\|i1}* = \frac{(\mathbf{a}'_{i2} \cdot \mathbf{a}^*_{i1})}{\mathbf{a}^2_{i1}} \mathbf{a}^*_{i1}$.
- Subtract to obtain $\mathbf{a}^*_{i2} = \mathbf{a}'_{i2\perp i1*} = \mathbf{a}'_{i2} - \mathbf{a}'_{i2\|i1}$.

1.1 Numerical Example

To illustrate Gram-Schmidt orthogonalization consider the following example of an arbitrary 3×3 full rank numerical matrix.

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \equiv \begin{bmatrix} 5 & 9 & 6 \\ 7 & -4 & -1 \\ 2 & 3 & -8 \end{bmatrix}$$

Although any of the three column vectors could have been chosen to provide (first) principal axis direction numbers, vector $\mathbf{a}^*_{i0} = \mathbf{a}_{i0}$ was selected and $\mathbf{a}_{i1}$ will be conditioned to provide $\mathbf{a}^*_{i1}$, the next.

$$\mathbf{a}_{i1\|i0} = \frac{1}{5^2+7^2+2^2}\left(\begin{bmatrix} 9 \\ -4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix}\right)\begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} = \frac{45-28+6}{78}\begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} = \frac{23}{78}\begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} = \frac{1}{78}\begin{bmatrix} 115 \\ 161 \\ 46 \end{bmatrix}$$

To maintain vectors in integer form the following difference can be multiplied through by 78.

$$\mathbf{a}^*_{i1} = \mathbf{a}_{i1\perp i0} = 78\begin{bmatrix} 9 \\ -4 \\ 3 \end{bmatrix} - \begin{bmatrix} 115 \\ 161 \\ 46 \end{bmatrix} = \begin{bmatrix} 702 & -115 \\ -312 & -161 \\ 234 & -46 \end{bmatrix} = \begin{bmatrix} 587 \\ -473 \\ 188 \end{bmatrix}$$

This was a one step process because the component $\mathbf{a}^*_{i1} = \mathbf{a}_{i1\perp i0}$ was chosen to provide direction numbers of the second principal Cartesian axis perpendicular to $\mathbf{a}^*_{i0} = \mathbf{a}_{i0}$. The next and final process involves two steps because the third orthogonal basis vector is found by first subtracting the component of $\mathbf{a}_{i2}$ that is parallel to $\mathbf{a}^*_{i0}$ to produce $\mathbf{a}'_{i2} = \mathbf{a}_{i2\perp i0}$ and then the component of $\mathbf{a}'_{i2}$ that is parallel to $\mathbf{a}^*_{i1}$ is subtracted from $\mathbf{a}'_{12}$ to produce $\mathbf{a}^*_{i2}$ that is normal to both $\mathbf{a}^*_{i0}$ and $\mathbf{a}^*_{i1}$.

$$\mathbf{a}_{i2\|i0} = \frac{1}{78}\left(\begin{bmatrix} 6 \\ -1 \\ -8 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix}\right)\begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} = \frac{30-7-16}{78}\begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} = \frac{1}{78}\begin{bmatrix} 35 \\ 49 \\ 14 \end{bmatrix}$$

$$\mathbf{a}'_{i2} = \mathbf{a}_{i2\perp i0} = 78\begin{bmatrix} 6 \\ -1 \\ -8 \end{bmatrix} - \begin{bmatrix} 35 \\ 49 \\ 14 \end{bmatrix} = \begin{bmatrix} 468 & -35 \\ -78 & -49 \\ -624 & -14 \end{bmatrix} = \begin{bmatrix} 433 \\ -127 \\ -638 \end{bmatrix}$$

$$\mathbf{a}_{i2\|i1^*} = \frac{1}{587^2+473^3+188^2}\left(\begin{bmatrix} 433 \\ -127 \\ -638 \end{bmatrix} \cdot \begin{bmatrix} 587 \\ -473 \\ 188 \end{bmatrix}\right)\begin{bmatrix} 587 \\ -473 \\ 188 \end{bmatrix}$$

$$\frac{254171+60071-119944}{603642}\begin{bmatrix} 587 \\ -473 \\ 188 \end{bmatrix} = \frac{194289}{603642}\begin{bmatrix} 587 \\ -473 \\ 188 \end{bmatrix} = \frac{1}{603642}\begin{bmatrix} 114052926 \\ -91902954 \\ 36528024 \end{bmatrix}$$

$$\mathbf{a}^*_{i2} = \mathbf{a}'_{i2} - \mathbf{a}_{12\|i1*} = 603642\begin{bmatrix} 433 \\ -127 \\ -638 \end{bmatrix} - \begin{bmatrix} 114052926 \\ -91902954 \\ 36528024 \end{bmatrix}$$

$$= \begin{bmatrix} 261376986 & -114052926 \\ -76662534 & +91902954 \\ -385123596 & -36528024 \end{bmatrix} = \begin{bmatrix} 147324060 \\ 15240420 \\ -421651620 \end{bmatrix}$$

Dividing these direction numbers by 2340 yields

$$\mathbf{a}^*_{i2} = \begin{bmatrix} 62959 \\ 6515 \\ -180193 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^* = \begin{bmatrix} 5 & 587 & 62959 \\ 7 & -473 & 6513 \\ 2 & 188 & -180193 \end{bmatrix}$$

Imposing a Euclidean norm produces the three unit column vectors in

$$\mathbf{A}_n^* = \begin{bmatrix} \frac{5}{\sqrt{78}} & \frac{587}{\sqrt{603642}} & \frac{29}{\sqrt{7739}} \\ \frac{7}{\sqrt{78}} & -\frac{473}{\sqrt{603642}} & \frac{3}{\sqrt{7739}} \\ \frac{2}{\sqrt{78}} & \frac{94\sqrt{2}}{\sqrt{301821}} & -\frac{83}{\sqrt{7739}} \end{bmatrix}$$

It is easy to show that

$$|\mathbf{A}_n^*| = 1 \text{ and } [\mathbf{A}_n^*][\mathbf{A}_n^*]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.2 Orthogonal Matrix as Rotational Operator

Examine Fig. 1.

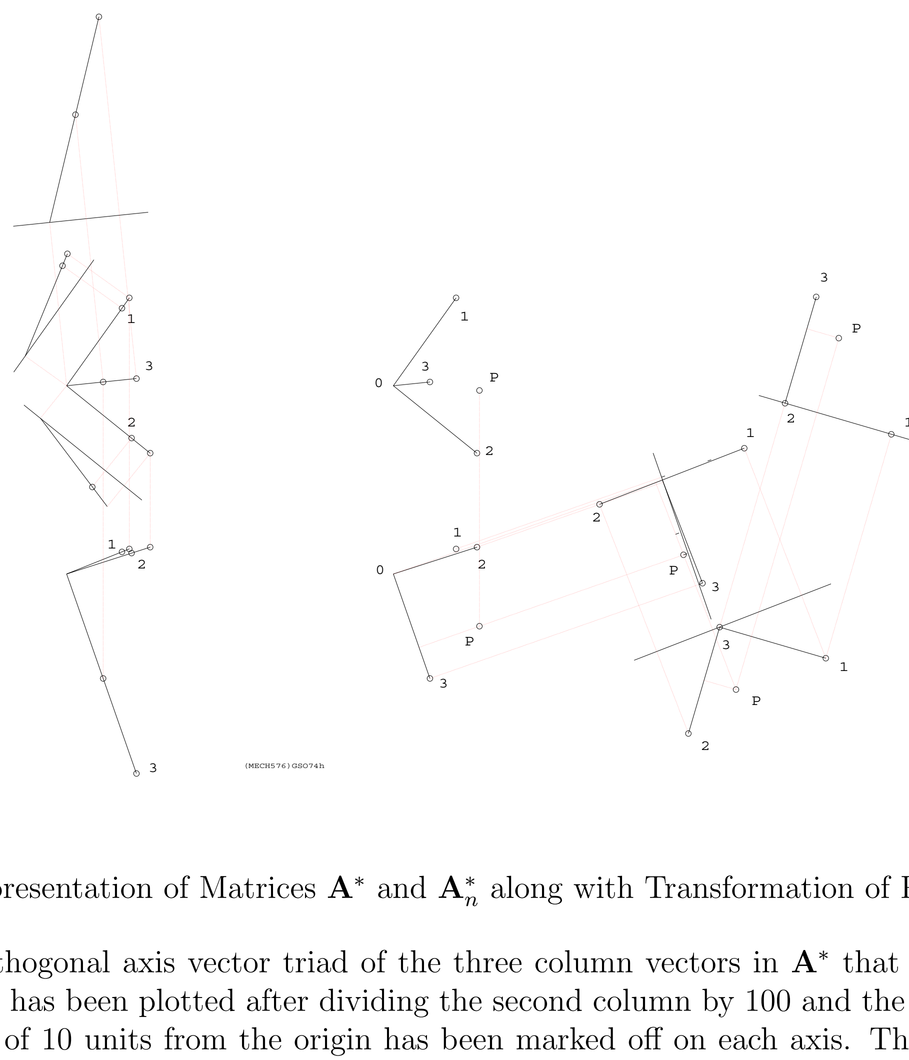

Figure 1: Representation of Matrices $\mathbf{A}^*$ and $\mathbf{A}_n^*$ along with Transformation of Point $P(3, 5, 7)$

At left, the orthogonal axis vector triad of the three column vectors in $\mathbf{A}^*$ that radiates from a common origin has been plotted after dividing the second column by 100 and the third by 10000. Then a length of 10 units from the origin has been marked off on each axis. This "normalized" frame is reproduced at the centre of the figure. Finally, via three auxiliary projections, two conjugate views of the frame have been produced so that one may take the vector 1 as x-axis, 2 as y-axis and 3 as z-axis. The view at upper right is a "front" view and its partner at lower right is an "under-side" view of this right handed Cartesian frame in which the point $P(3, 5, 7)$ has been plotted. When this point is projected back to the conjugate view pair in the middle of the figure, where the three column vectors of $\mathbf{A}_n^*$ were plotted in top and front views, one measures its coordinates as, approximately, $P(7.784, -0.427, -4.715)$, the same result is obtained with the following matrix multiplication.

$$\begin{bmatrix} \frac{5}{\sqrt{78}} & \frac{587}{\sqrt{603642}} & \frac{29}{\sqrt{7739}} \\ \frac{7}{\sqrt{78}} & -\frac{473}{\sqrt{603642}} & \frac{3}{\sqrt{7739}} \\ \frac{2}{\sqrt{78}} & \frac{94\sqrt{2}}{\sqrt{301821}} & -\frac{83}{\sqrt{7739}} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 7.783599778 \\ -0.4274821649 \\ -4.715170571 \end{bmatrix}$$

2 Upper Triangularization and Matrix Rank

The numerical matrix used in the orthogonalization example above will now be used to illustrate that a full rank square matrix will contain no rows with all zero elements, *i.e.*, the diagonal elements are all non-zero, after all elements beneath the diagonal $a_{01}, a_{12}, \ \ldots \ a_{n-1,n}$ have been zeroed by Gaussian elimination.

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} a''_{00} & a''_{01} & a''_{02} \\ 0 & a''_{11} & a''_{12} \\ 0 & 0 & a''_{22} \end{bmatrix}$$

$$\begin{bmatrix} 5 & 9 & 6 \\ 7 & -4 & -1 \\ 2 & 3 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 9 & 6 \\ 0 & -83 & -47 \\ 0 & -3 & -52 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 9 & 6 \\ 0 & -83 & -47 \\ 0 & 0 & 4175 \end{bmatrix}$$

The original 3×3 numerical example matrix was upper triangularized by the following sequence of subtractions.

$$\mathbf{a}''_{1j} = 5\mathbf{a}_{0j} - 7\mathbf{a}_{1j}$$

$$\mathbf{a}^{\#}_{2j} = 5\mathbf{a}_{2j} - 2\mathbf{a}_{0j}$$

$$\mathbf{a}''_{2j} = -83\mathbf{a}^{\#}_{2j} - (-3\mathbf{a}''_{1j})$$

It is obvious that this is not an orthogonal matrix. It would be diagonal if it were. If normalized, it would be a unitary matrix.

2.1 Upper Triangularization and Rank Deficiency

Consider the matrix below.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -5 & -4 & -3 \end{bmatrix}$$

Its third row has been deliberately constructed as

$$\mathbf{a}_{2j} = 3\mathbf{a}_{0j} - 2\mathbf{a}_{1j}$$

Subtracting $\mathbf{a}_{1j} - 4\mathbf{a}_{0j}$ and $\mathbf{a}_{2j} - (-5\mathbf{a}_{0j})$ produces

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 6 & 12 \end{bmatrix}$$

It is now obvious that elements of the last row are in the same ration as those of the second. The next elimination step produces

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

so the rank of this matrix is two.

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