MECH 314
Dynamics of Mechanisms

February 17, 2011

Offset Slider Crank Analysis and Some Other Slider Systems

1 Offset Slider Crank Position Analysis

Position and velocity analysis for the *common* slider-crank has already been dealt with. That is a special case, where the crank centre point was taken as origin O, that corresponds to the point A of our "standardized" labeling for planar 4-bar linkages. In Fig. 1 one sees a more general slider-crank mechanism arrangement. The common one is produced by choosing the line of wrist-pin excursion to be on $O = A$. The equation of the line thus becomes $y = 0$ because the offset $k = 0$. Notice that the crank radius has been normalized to $s = 1$ with crank-pin at point D. Point C is on the wrist-pin and the point B can be imagined as located at an indefinitely large distance in a direction normal to $y + k = 0$. So there is no point in trying to label point B or links $AB = l$ and $BC = p$.

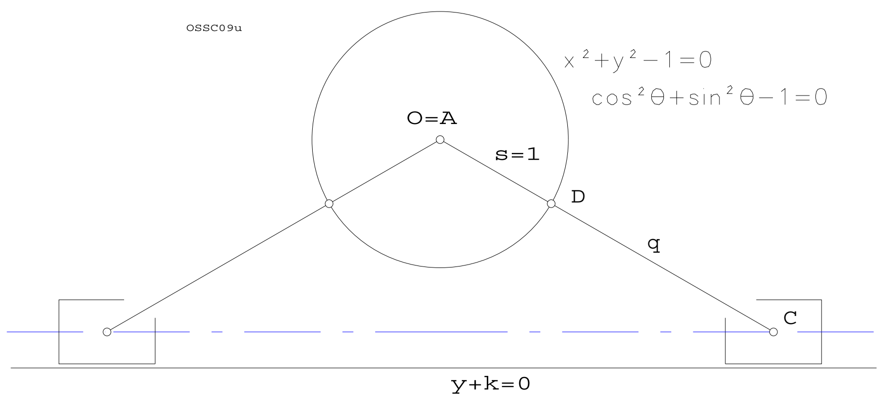

Figure 1: Offset Slider Crank at Extended Piston Travel in Both Branches

A position analysis is deemed necessary for two reasons.

- It is easy to draw the two singular configurations shown. One merely intersects the following circle and line

$$x^2 + y^2 - (q + s)^2 = x^2 + y^2 - (q + 1)^2 = 0, \;\; y + k = 0$$

  but is it certain that C does not move further to the right –or left in the other branch– as the crank rotates to approach the horizontal?

- Furthermore how far will C retract, along $y + k = 0$, so as to be more nearly under $O = A$, while the motion remains in the same *assembly mode* or branch? This question is complicated by the fact that if D is reflected by π through A the connecting rod CD is obviously not conveniently on the line ADC in this position so there is no "three-R-joint-centres-in-a-row" indication of singularity.

- Moreover we an indulge in a little "optimization" exercise.

The equation of any circle of radius q and centred on D, as the crank is rotated, and then intersected with the line on which C travels is written

$$(x - x_D)^2 + (y - y_D)^2 - q^2 \rightarrow (x - \cos\theta)^2 + (y - \sin\theta)^2 - q^2 \rightarrow (x - \cos\theta)^2 + (k - \sin\theta)^2 - q^2 = 0$$

Making the conventional tangent half-angle substitution of $u = \tan\frac{\theta}{2}$, where θ is the CCW rotation of the crank measured from the positive x-axis, and multiplying out the denominator $(1+u^2)$ produces a bivariate quadratic in x and u. The form of this polynomial $f(u,x)$, collected on u, is written below.

$$f = \left[(x+1)^2 + k^2 - q^2\right]u^2 + 4ku + \left[(x-1)^2 + k^2 - q^2\right] = 0$$

Since we are interested in critical points of extreme piston excursion of the function $f = f(u,x)$ it is the derivative

$$f' = \frac{dx}{du} = -\frac{\partial f/\partial u}{\partial f/\partial x} = 0$$

that must vanish. The minus sign is of no consequence and neither is the denominator so long as *it* does not vanish. So we are looking for the simultaneous solution of $f = 0$ and $f' = 0$ to find critical values of u. The numerator derivative that must vanish is

$$\frac{\partial f}{\partial u} = \left[(x+1)^2 + k^2 - q^2\right]u + 2k = 0$$

This leads, upon eliminating x, to a quartic in u.

$$k^2u^4 + 4ku^3 + 2(k^2 - 2q^2 + 2)u^2 + 4ku + k^2 = 0$$

Amazingly, for the example shown in Fig. 2, there are, in the line below, four *real* roots that are tangents of half the crank-angles θ at he limits of wrist-pin excursion x.

k	q	$D'- \equiv \tan\frac{36.8699°}{2}$	$D- \equiv \tan\frac{143.1301°}{2}$	$D+ \equiv \tan\frac{-25.3769°}{2}$	$D'+ \equiv \tan\frac{-154.6230°}{2}$
3	6	$1/3$	3	-0.225148226	-4.4441518440

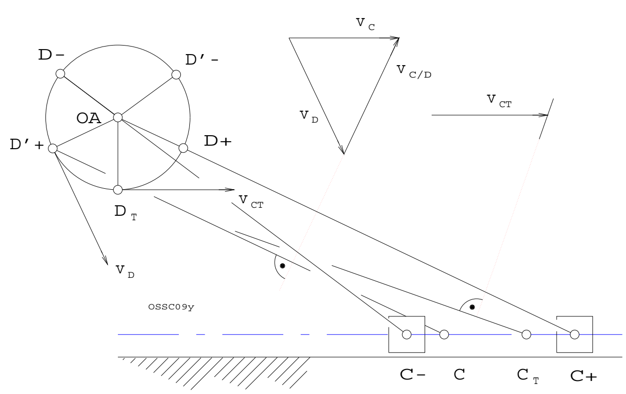

Figure 2: Limits, $C+$ and $C-$, of Travel in Right Branch

2 Two Alternate Approaches

As satisfying as the results tabulated above may appear some may quail at the prospect of procedures that involve

- The change of variable from θ to u,
- Followed by extracting the total derivative of an implicit function,
- Then simultaneous solution of differential and displacement equations and
- Algebraic manipulation necessary to get the neat univariate quartic in u.

Some of the steps are difficult to do without a symbolic math package like *Maple* or *Mathematica*.

2.1 Derivative of the Explicit Displacement Equation

Possibly the most direct approach is to begin with the line/circle intersection equation

$$(x - \cos\theta)^2 + (-k - \sin\theta)^2 - q^2 = 0$$

and solve quadratically for x. Taking the positive root, that represents the right-hand branch, the derivative is

$$\frac{dx}{d\theta} = \frac{\sin\theta\sqrt{\cos^2\theta - 2k\sin\theta - 1 - k^2 + q^2} + \cos\sin\theta + k\cos\theta}{\sqrt{\cos^2\theta - 2k\sin\theta - 1 - k^2 + q^2}}$$

Seeking maxima and minima of this function with respect to θ requires setting the numerator to zero hoping that the denominator is not zero. But the square-root term is annoying so we can separately square it and the term $\cos\sin\theta + k\cos\theta$ and form the difference of these two squares. After, again, some trigonometric simplification we get a nice quadratic in $\sin\theta$.

$$(1 - q^2)\sin^2\theta + 2k\sin\theta + k^2 = 0$$

With $k = 3$ and $q = 6$ the two roots of this equation are $-3/7$ and $3/5$ and the four angles that correspond to these sines are the ones previously tabulated. The pitfalls of this method include "finger-trouble" in taking the derivative and in trigonometric manipulations. Nevertheless it is believed most students would prefer to do it this way rather than using the previous or next method. It is emphasized here that, although we present the complete solutions to *this* problem in detail, the idea is to equip you to tackle related but somewhat different problems and to help you learn to think *geometrically*.

2.2 A Geometric Approach

The magic geometric word here is *singularity* or three-points-in-a-line. Recall how that was the key to the other complicated problem we did concerning time ratio. Finding the maximum wrist-pin excursion at $C+$ and its crank angle at $D+$ was easy. We just drew a circle of radius $q + s$ centred on $0 = A$ and intersected it with the line $y + k = 0$. $C-, D-$ is not so easy but we can express colinearity of ACD and the intersecting circle radius, $q - 2s$ in this case. Since two points define a line, three linearly independent points in the plane subtend a finite triangular area. If the three are in a line the area vanishes. This is expressed by the determinant of the matrix whose rows are the homogeneous coordinates of D, C, A.

$$-\left|\begin{bmatrix} 1 & \cos\theta & \sin\theta \\ 1 & x & -k \\ 1 & 0 & 0 \end{bmatrix}\right| = k\cos\theta + x\sin\theta = 0$$

Now we express the square of the distance from D to C as q^2.

$$(cos\theta - x)^2 + (sin\theta + k)^2 - q^2 = 0$$

It is easy to eliminate x between these two equations because it is linear in the first. Again, with some trigonometric simplification, we obtain the equation

$$(1 - q^2)\sin^2\theta + 2k\sin\theta + k^2 = 0$$

which is identical to the previous quadratic univariate in $\sin\theta$.

3 Conclusion

On Fig. 2 points $D'+$ and $D'-$ are the crank-pin positions for the left branch assembly mode that respectively correspond to the farthest and closest excursions of the wrist-pin at $C'+$ and $C'-$ (not shown). Position $D'+$ was taken arbitrarily to illustrate computation of velocities, graphically. *E.g.*, given, say, ω_s one may compute the crank-pin velocity $v_D = s\omega_s$, taken, of course, in the correct sense as given, CW or as in this case CCW, so the velocity vector $\mathbf{v}_D$ is normal to $\mathbf{r}_{AD}$. The velocity $\mathbf{v}_{C/D}$ is normal to the connecting rod $q = DC$ and the velocity v_C must be horizontal and must vectorially satisfy $\mathbf{v}_C = \mathbf{v}_D + \mathbf{v}_{C/D}$ as shown. The other construction at D_T shows a position where the angular velocity ω_q of the connecting-rod must be zero because the $\mathbf{v}_T$ must be identically horizontal at both ends of the connecting rod.

4 Problem P3.24

Examine Fig. 3. This is, when inverted by fixing the rod AB, an ordinary slider crank –not even offset– with A as the crank centre, OD becomes the connecting rod with D becoming the slider wrist pin. The problem at hand is

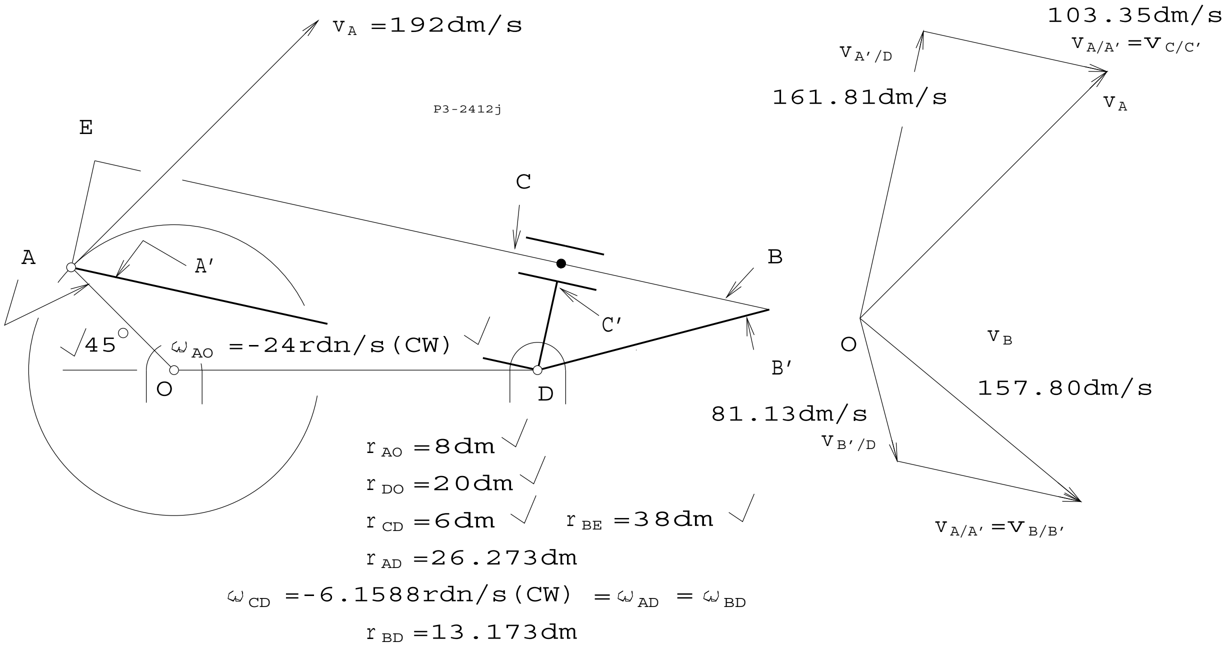

Figure 3: An Inverted Slider Crank

to do a complete velocity analysis given the five parameters indicated with ✓. In a nutshell, the *direct* approach, *i.e.*, without resorting to inversion, is to imagine the three links DA', DC' and DB, shown in bold, moving as one, rotating about fixed point D. In fact only DC' exists. Clearly A' will not match the velocity of A, rotating about fixed centre O, unless there is sliding of BCA, all together, in a direction parallel to BC. With this insight in hand one can do the graphical velocity analysis shown in Fig. 3 or the following vector analysis. As is often the case, finding the displacement vectors between points is tedious and prone to making slips. First the angular velocity $\omega_{AD} = \omega_{CD} = \omega_{BD}$ of the three links and the relative sliding velocity $\mathbf{v}_{A/A'} = \mathbf{v}_{B/B'} = \mathbf{v}_{C/C'}$ and then the velocity $\mathbf{v}_B$ are found.

$$
\begin{aligned}
\boldsymbol{\omega}_{AO} \times \mathbf{r}_{OA} - \boldsymbol{\omega}_{AD} \times \mathbf{r}_{DA} - \mathbf{v}_{C/C'} &= \mathbf{0} \\
-\omega_{AO} r_{OAy} + \omega_{AD} r_{ADy} - k_{CC'} r_{DAx} &= 0 \\
\omega_{AO} r_{OAx} - \omega_{AD} r_{ADx} - k_{CC'} r_{DAy} &= 0 \\
\mathbf{v}_B + \boldsymbol{\omega}_{AD} \times \mathbf{r}_{DB} - \mathbf{v}_{B/B'} & \\
v_{Bx} + \omega_{AD} r_{DBy} - k_{CC'} r_{DAx} &= 0 \\
v_{By} - \omega_{AD} r_{DBx} - k_{CC'} r_{DAy} &= 0
\end{aligned}
$$

Results shown in Fig. 3 are also obtained by substituting the following numerical values for the various parameters. $\varphi = \angle ODA$

$$r_{OAx} = -8/\sqrt{2}, \;\; r_{OAy} = -r_{OAx}, \;\; r_{DAx} - 20 - 8/\sqrt{2}, \;\; r_{DAy} = r_{OAy}$$

$$r_{DBx} - 8/\sqrt{2} + 6\sin\varphi + 38\cos\varphi - 20, \;\; r_{DBy} = 8/\sqrt{2} + 6\cos\varphi - 38\sin\varphi$$

$$h = \sqrt{r_{DAx}^2 + r_{DAy}^2}, \;\; \sin\varphi = r_{DAy}/h, \;\; \cos\varphi = -r_{DAx}/h, \;\; \omega_{AO} = -24\text{rdn/s(CW)}$$

5 Problem P3.25

The trick in doing the velocity analysis of the problem shown in Fig. 4 is to formulate the velocity $\mathbf{v}_{C'}$ so that it is in the direction of the line BD so as not interfere with but slide smoothly in the tilting slider block. Once more

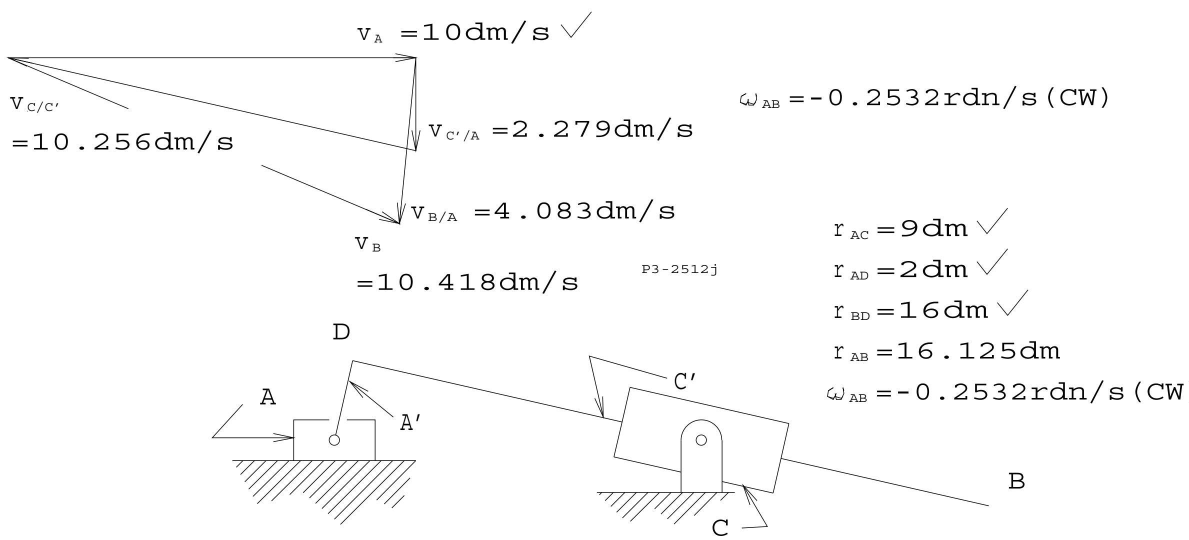

Figure 4: "Threading a Needle?"

the items marked ✓are given. A complete graphical velocity analysis is provided above Fig. 4. A vector analytical approach is presented in the equations below.

$$\mathbf{v}_A + \boldsymbol{\omega}_{AC} \times \mathbf{r}_{AC} - \mathbf{v}_{C'/C} = \mathbf{0}$$
$$v_{Ax} - \omega_{AC} r_{ACy} - k_C r_{CDx} = 0, \;\; v_{C'/Cx} = k_C r_{CDx}$$
$$v_{Ay} + \omega_{AC} r_{ACx} - k_C r_{CDy} = 0, \;\; v_{C'/Cy} = k_C r_{CDy}$$

The numerical substitutions required to obtain the answers listed in Fig. 4 are as follows. $\varphi = \angle ACD$.

$$r_{ACx} = 9, \;\; r_{ACy} = 0, \;\; \cos\varphi = \sqrt{77}/9, \;\; \sin\varphi = 2/9, \;\; r_{CDx} = -\sqrt{77}\cos\varphi, \;\; r_{CDy} = \sqrt{77}\sin\varphi$$
$$r_{ABx} = 4/9 - 16 r_{CDx}/\sqrt{77}, \;\; r_{ABy} = 2\sqrt{77}/9 - 16 r_{CDy}/\sqrt{77}, \;\; v_{Ax} = 10, \;\; v_{Ay} = 0$$

6 Problem P3.45

This problem is interesting because its motion is dictated by relationship between the angle θ through which the gear OB turns, the angle $\varphi = \angle BAO$ of the the rack and the distance of up/down slider travel of point A. Notice that A can move up only to $y = -R$ where $\varphi = \pi/2$. R is the gear pitch radius. Although we won't use first kinematic coefficient this exercise allows one to write three equations relating θ, φ, y.

$$(\theta^2 + 1)\sin^2\varphi - 1 = 0, \;\; y\sin\varphi - R = 0, \;\; R^2(\theta^2 + 1) - y^2 = 0$$

Taking sums of partial derivatives with respect to θ, φ, y as these apply one obtains

$$\theta\sin^2\varphi + (\theta^2 + 1)\sin\cos\varphi = 0, \;\; y\cos\varphi + \sin\varphi = 0, \;\; R^2\theta - y = 0$$

Adding the angular and linear velocity time derivatives

$$\theta\sin^2\varphi\dot{\theta} + (\theta^2 + 1)\sin\cos\varphi\dot{\varphi} = 0, \;\; y\cos\varphi\dot{\varphi} + \sin\varphi = 0, \;\; R^2\theta\dot{\theta} - y\dot{y} = 0$$

Simplifying the first two of these

$$\theta \tan\varphi \dot{\theta} + (\theta^2 + 1)\dot{\varphi} = 0, \quad y\dot{\varphi} + \tan\varphi \dot{y} = 0$$

Substituting $y = -0.4, \quad \tan\varphi = 1/\sqrt{3}, \quad \dot{y} = 1.5$ yields $\dot{\varphi} = \omega_{AB}$ of the graphical solution in Fig. 5. Substituting

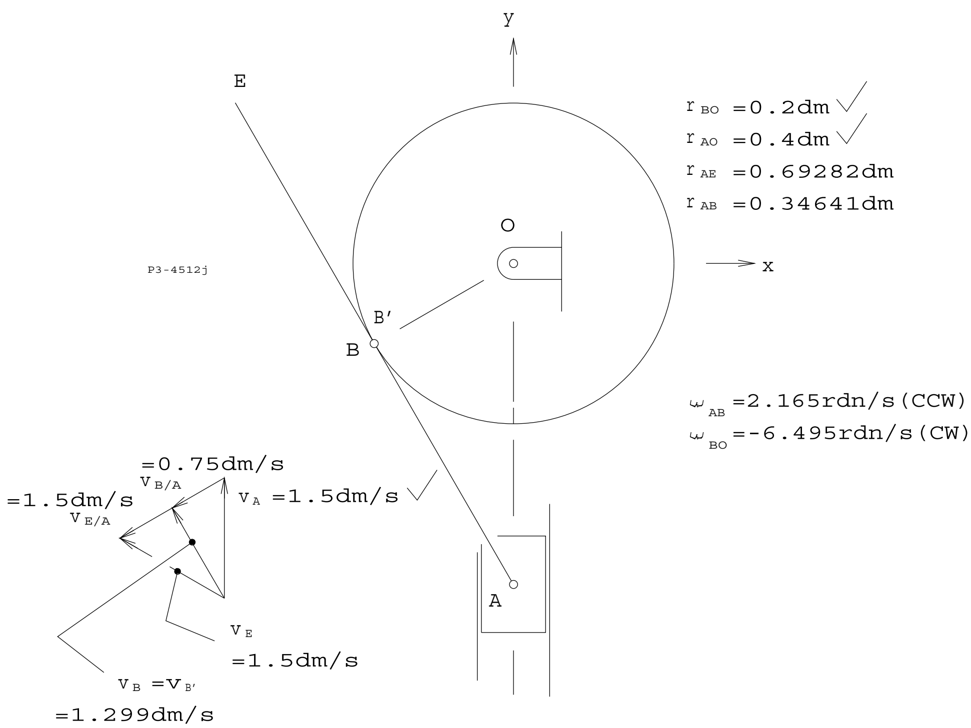

Figure 5: Gear, Tilting Rack and Slider

$R = 0.2, \quad y = -0.4$ yields $\theta = 1.73205$rdn. Finally substituting $R = 0.2, \quad y = -0.4, \quad \dot{y} = 1.5$ yields $\dot{\theta} = -8.66025$rdn/s $\neq \omega_{BO}$ in the graphical solution. Can you help?

6.1 Resolving the Dilemma

The graphical solution, indeed, produces the correct answer. Why? The little thought experiment shown in Fig. 6 will explain why. It all boils down to having overlooked an important element of the displacement kinematics so the differentiation produced a faulty velocity expression. The correct expression for the angle θ through which the gear turns is

$$\theta = \frac{1}{R}\sqrt{y^2 - R^2} - \cos^{-1}\frac{R}{-y}$$

One must subtract an angle from the rack-string-pull-off model that was originally formulated. In the leftmost figure one sees the the rack correctly placed at $\theta = 0$, its right hand tip is A, coincident with the bottom of the gear. Pulling the rack to the right a distance $\sqrt{y^2 - R^2}$, while maintaining it in its horizontal position, causes the gear to rotate CCW so the original contact point P on the gear, originally coincident with A on the rack, is now in the position shown in the centre diagram. Now rack and gear are in contact at B so it must be kept there. Finally the gear and rack, treated as a rigid body unit, is rotated CW about O the gear centre so A and B assume their rightful position as shown in the right hand diagram. Taking the time derivative $\frac{d\theta}{dt}$

$$\dot{\theta} = \left[\frac{y}{R^2\sqrt{\frac{y^2-R^2}{R^2}}} + \frac{R}{y^2\sqrt{1 - \frac{R^2}{y^2}}}\right]\dot{y}$$

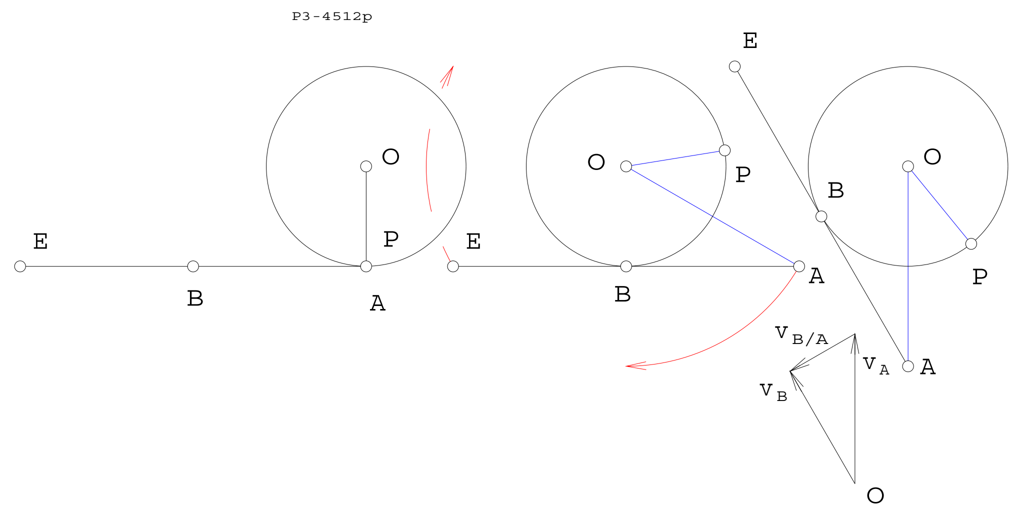

Figure 6: A Thought Experiment to Show Angle $\theta = \angle AOP$

Now when one substitutes $R = 20,\ y = -40,\ \dot{y} = 150$ into this expression the correct result $\dot{\theta} = -6.4952$rdn/s is obtained and the angle shown in Fig. 6 is indeed that obtained with the equation for θ, *i.e.*, $\theta = 39.24°$CCW. Now we must go back and revise the first and last in each of the three sets of three equations in this section, *i.e.*,

$$f(\varphi, \theta),\ \ g(y, \theta),\ \ \frac{\partial f}{\partial \varphi} + \frac{\partial f}{\partial \theta},\ \ \frac{\partial g}{\partial y} + \frac{\partial g}{\partial \theta},\ \ \dot{f}(\varphi, \theta),\ \ \dot{g}(y, \theta)$$