Geometric Construction Problem 2:- A Connection

(MECH289)MC61f.tex

January 10, 2006

1 Problem:- Assigned 06-02-10, Due 06-02-17

Read this document and repeat the computation necessary to locate the end points K' and L' on line segments AB and CD as shown in Fig. 1. In that drawing the connection KL is shown, using the direction as specified KL on the "Monge's sphere" on the right and the problem is solved with descriptive geometric construction. Your job is to find the *other* connection. Hand in your work, consisting of some neat freehand sketches and your calculations, to Aya or Vahid before Monday 06-02-17. Show me that you are capable of "geometric thinking" and win *up to* 5% of your final grade before we even start.

2 Descriptive Geometry

As you can see, the required connection is specified as -must be- normal to line CD and make an angle of 45° where it joins AB. Read notes **"Fundamentals of Geometric Construction"**, section **7 Joining Line Problems**, pp.66-71. The construction procedure used is the general one. First, the *direction* of the required connection was established. Then the plane segment ABE was constructed to contain AB and in an attitude parallel to the direction KL. K is where the line CD intersects the plane ABE. To get L, a line was drawn from K on CD, parallel to the required direction, and extended to intersect AB. To find out how to construct a plane segment on a given line segment and parallel to another line you will have to spend some time struggling through sections **2.1 - 2.8**, pp.2-20. If you have survived the experience, you will also have learned how to construct the intersection between line and plane!

3 Analytical Geometry

But this, alas, not a geometry course. If you are interested in one, you may try MECH 576, later. Instead, use Eq. (32) on p.71 to find the direction. Then look at Fig. 8 and solve Eqs. (1) and (2) on p.19.

3.1 Grassmannian Determinants and Plücker Coordinates

You may ignore this subsection entirely. Nevertheless, classical geometry provides a neat way to do this problem and opportunity to illustrate some aspects of "geometric thinking". To find the direction KL on Monge's sphere we proceed by observing that if a point on the sphere, common to the two cones with their apeces on the sphere centre, is found, then the required direction is established by joining that point to the sphere centre. This can be done by finding the intersection of a line with the sphere. To learn about Monge's construction to find the common directions between two pencils of lines that respectively rule two cones of revolution so as to make specified

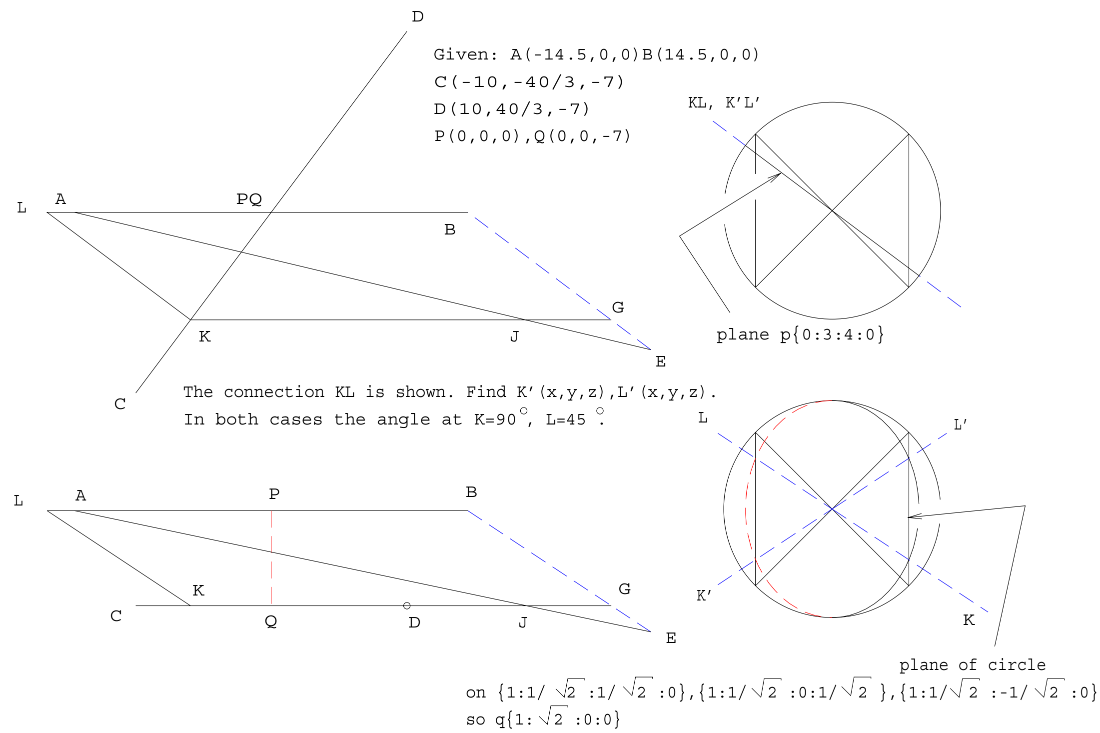

Figure 1: One of Two Possible Connecting Lines

angles with respect to the cone axes, study Fig. 4. These axes are respectively parallel to the lines that must be so joined.

3.1.1 Two Planes

What line? A line on two planes. Notice that such cone-sphere intersections are circles and circles are planar curves. The normal connection at K requires a "cone" with an apex half-angle of 90°; a plane on the sphere centre, really. See it? Since this plane contains the sphere centre, chosen as a local origin, the plane equation has no "constant" coefficient and we write the equation coefficients as *homogeneous plane coordinates*. Since this plane is normal to CD, the *direction numbers* of CD provide the other three coordinates.

$$p\{0:3:4:0\}$$

The other plane, a circle on the sphere and the 45° cone that specifies line directions with respect to AB, is on three immediately available points if this is taken as a sphere of unit radius. Then that plane's equation is given by the Grassmannian determinant of the singular equation with rows consisting of the three sets of point coordinates, thus.

$$q:\begin{vmatrix} w & x & y & z \\ 1 & a & a & 0 \\ 1 & a & 0 & a \\ 1 & a & -a & 0 \end{vmatrix} = 0$$

With $a = 1/\sqrt{2}$, obviously, the determinant expands to give the following plane equation.

$$\begin{vmatrix} a & a & 0 \\ a & 0 & a \\ a & -a & 0 \end{vmatrix} w - \begin{vmatrix} 1 & a & 0 \\ 1 & 0 & a \\ 1 & -a & 0 \end{vmatrix} x + \begin{vmatrix} 1 & a & 0 \\ 1 & a & a \\ 1 & a & 0 \end{vmatrix} y - \begin{vmatrix} 1 & a & a \\ 1 & a & 0 \\ 1 & a & -a \end{vmatrix} z = 0$$

Expanding the four 3×3 determinants produces the plane equation, which, after substituting for a, setting $w = 0$ and simplifying, gives the following plane equation and its homogeneous coordinates.

$$1 - \sqrt{2}x = 0, \quad q\{1 : -\sqrt{2} : 0 : 0\}$$

3.1.2 Line and Sphere

A similar Grassmannian determinant expansion, this time of a doubly rank deficient matrix containing only two linearly independent *plane, not point* coordinate rows, produces, with six 2×2 determinants, the following six *Plücker* or *line* coordinates of line

$$\mathcal{G}\{G_{01} : G_{02} : G_{03} : G_{23} : G_{31} : G_{31}G_{12}\}$$

.

$$\mathcal{G} : \begin{vmatrix} w & x & y & z \\ w' & x' & y' & z' \\ 0 & 3 & 4 & 0 \\ 1 & -\sqrt{2} & 0 & 0 \end{vmatrix} = 0 \to \mathcal{G}\{-3 : -4 : 0 : 0 : 0 : 4\sqrt{2}\} \to \left\{\frac{3}{4} : 1 : 0 : 0 : 0 : -\sqrt{2}\right\}$$

The unit sphere u is given by

$$u : x^2 + y^2 + z^2 = 1$$

Now we impose the geometric condition that a point $S\{s_0 : s_1 : s_2 : s_3\}$ is simultaneously on $\mathcal{G}$ and u. This is satisfied if $\mathcal{G} \cap S$ defines no plane $s\{S_0 : S_1 : S_2 : S_3\}$.

$$S \in \mathcal{G}, \quad \sum_{j=0}^{3} G_{ij}s_j = S_i = 0, \quad \forall \;\; i = 0, 1, 2, 3$$

Note conventions that $G_{ij} = 0 \;\; \forall \;\; i = j$ and $G_{ji} = -G_{ij}$. The four sums are written below.

$$\begin{array}{rrrrl} & G_{01}s_1 & +G_{02}s_2 & +G_{03}s_3 & = S_0 = 0 \\ -G_{01}s_0 & & +G_{12}s_2 & -G_{31}s_3 & = S_1 = 0 \\ -G_{02}s_0 & -G_{12}s_1 & & +G_{23}s_3 & = S_2 = 0 \\ -G_{03}s_0 & +G_{31}s_1 & -G_{23}s_2 & & = S_3 = 0 \end{array}$$

Making appropriate substitutions for G_{ij} produces the following results.

$$x = s_1 = \frac{1}{\sqrt{2}}, \quad y = s_2 = -\frac{3}{4\sqrt{2}}$$

Finally the sphere equation gives

$$z = \pm\frac{\sqrt{14}}{8}$$

Now we have the three direction numbers of DE -note that plane segment CDE, instead of ABE, was used in Fig. 8- to define Eqs. (1) and (2) on p.19. In the calculations below **k,l,p,q** are position vectors of points K, L, P, Q where $P(0,0,0)$ is taken as origin. **s** is the direction number vector obtained from $x,, y, z$, above.

$$\mathbf{l} = \mathbf{p} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \mathbf{q} + u\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{k} - \mathbf{l} = v\mathbf{s}$$

t, u, v are scalar parameters and the following three equations yield required position vectors **k,l** after solving for t, u, v.

$$\begin{bmatrix} 0 \\ 0 \\ -7 \end{bmatrix} - \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3u \\ 4u \\ 0 \end{bmatrix} - \begin{bmatrix} 4v \\ -3v \\ -\sqrt{7}v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{k} = \begin{bmatrix} -\frac{9\sqrt{7}}{4} \\ -3\sqrt{7} \\ -7 \end{bmatrix}, \quad \mathbf{l} = \begin{bmatrix} -\frac{25\sqrt{7}}{4} \\ 0 \\ 0 \end{bmatrix}$$

4 A Solid Modelling Exercise

Now join two ducts of radius 2 units with two toroidal transition pieces with tangential continuity on a normal plane on the midpoint of segment $K'L'$. Report the coordinates of R', S', T', points on the normal planes of tangency on AB, KL and CD, respectively. Points R, S, T for connection KL are shown in Fig. 2. This is preparatory to creating the actual solid model. The result of *my* connection is shown in Fig. 3.

(MECH289)MC61f.tex

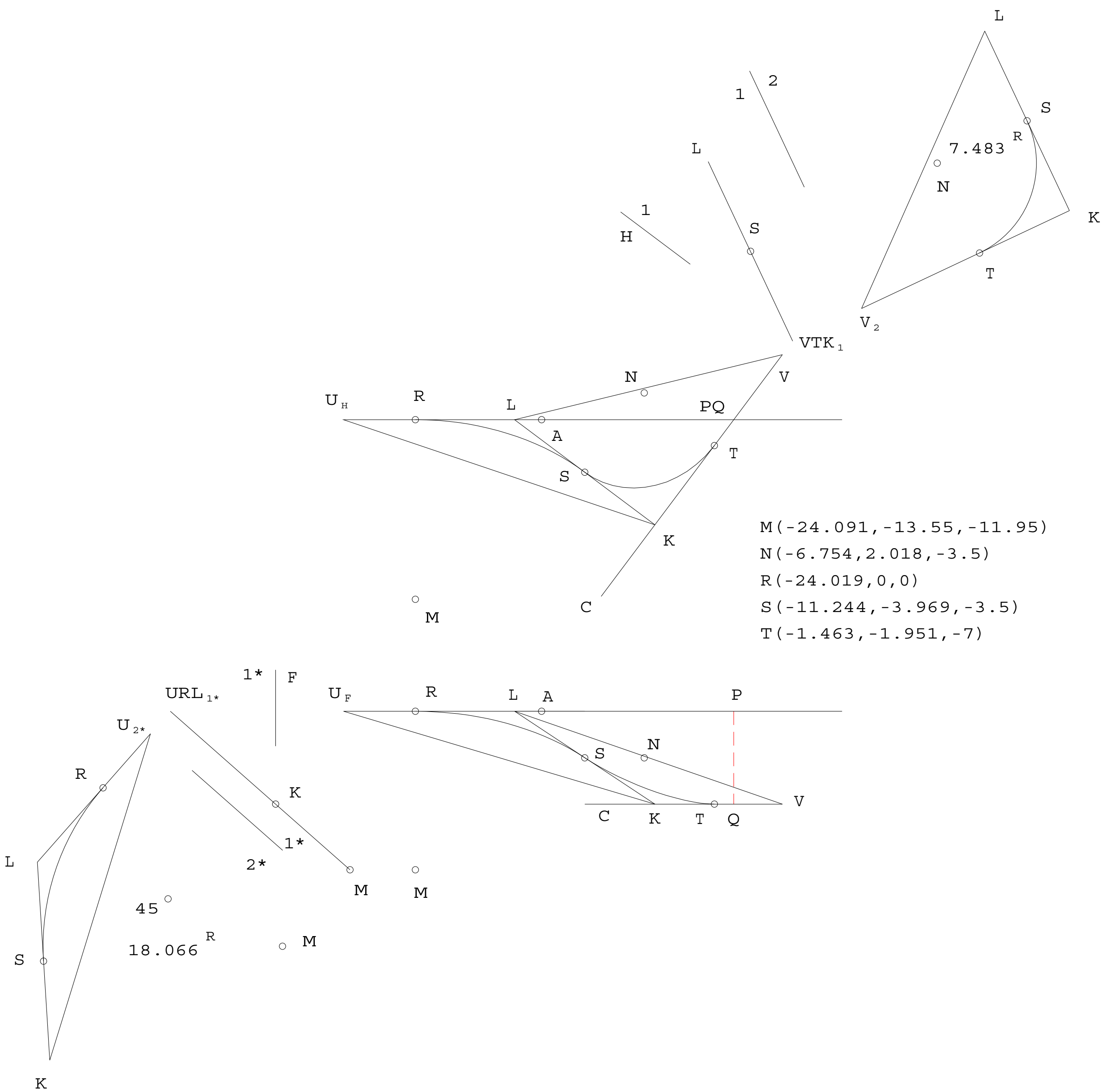

Figure 2: Finding Tangent Points R, S, T and Arc Centres

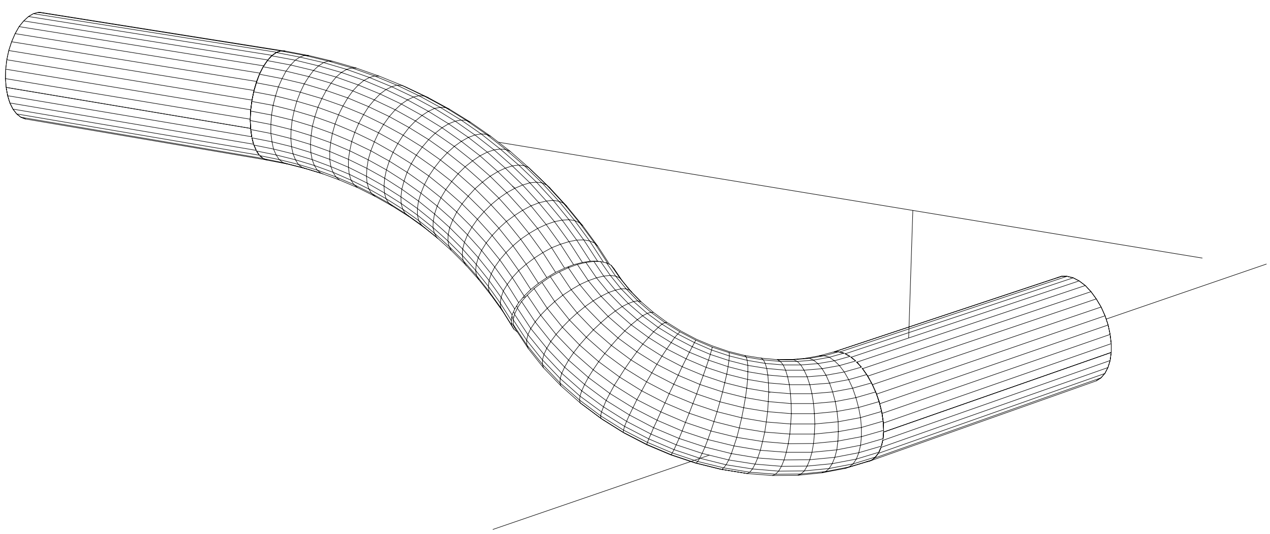

Figure 3: Solid Model of Connection between Two Ducts on Skew Centrelines

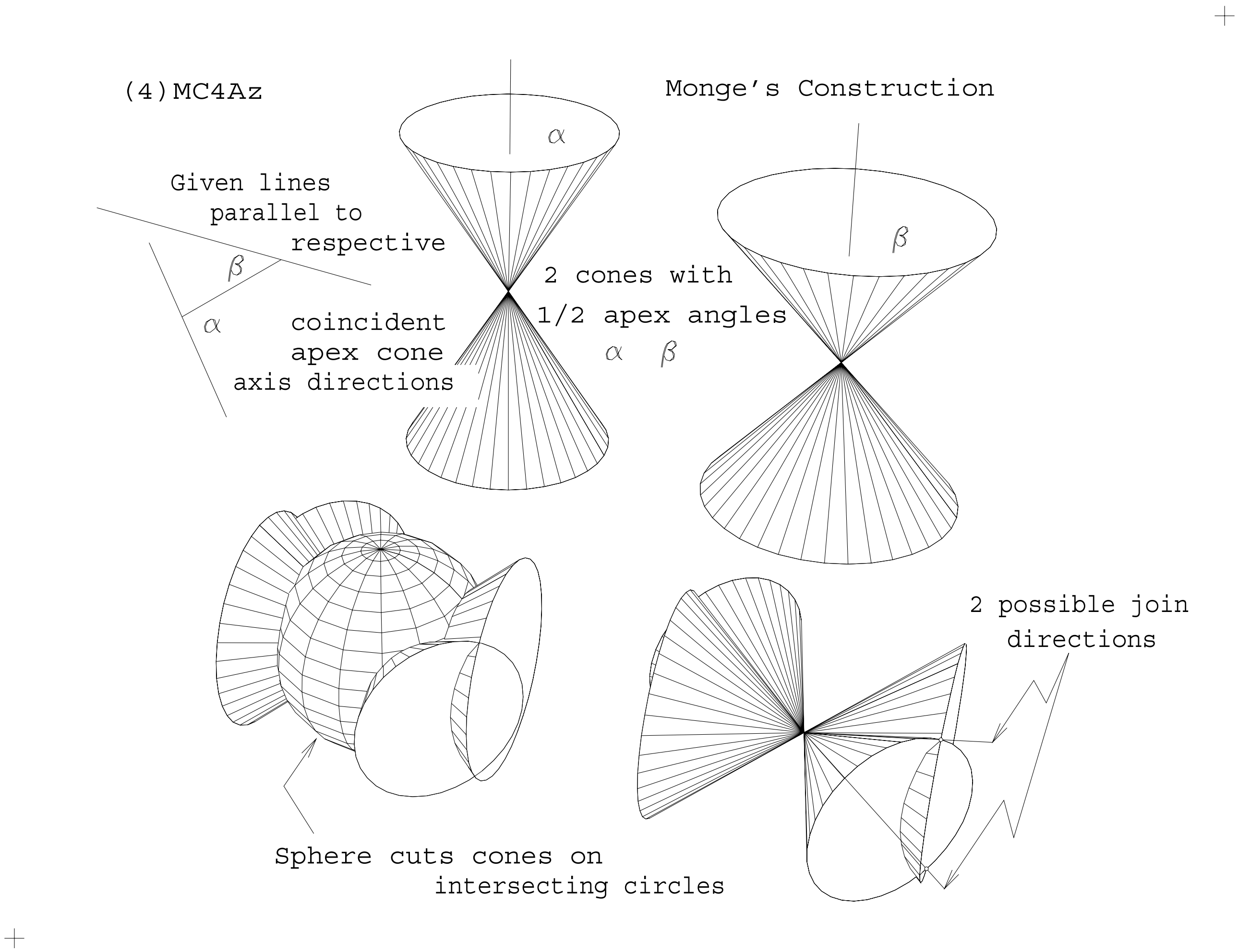

Figure 4: Visualizing Monge's Construction