MECH 576
Geometry in Mechanics

September 16, 2009

Line Geometry Primer

0. LINE GEOMETRY (what is it about?)

Writing constraint equations dealing with lines in 3D space

Writing efficient, test-free algorithms

Mantra: "Computers are fast, memory is cheap"

Answer: Not fast enough for real-time control, not cheap enough for large-scale simulation

1. LINE GEOMETRY (what is it?)

Imagine an axial force member, in MECHANICS (Statics), which may carry either tension (T) or compression (C).

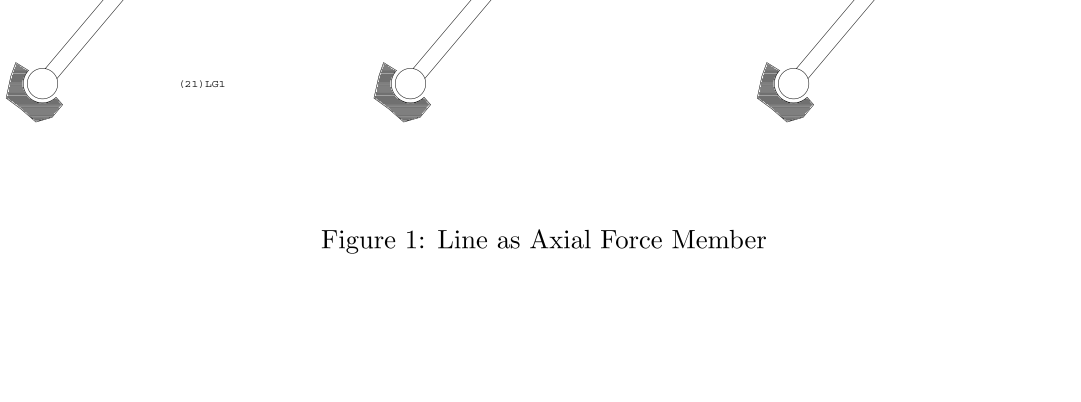

Figure 1: Line as Axial Force Member

2. DIRECTION & MOMENT

The axial force has a <u>direction</u> and a <u>moment</u>. Since a member may sustain a continuous range of load magnitudes, only the ratios among direction and moment components are relevant.

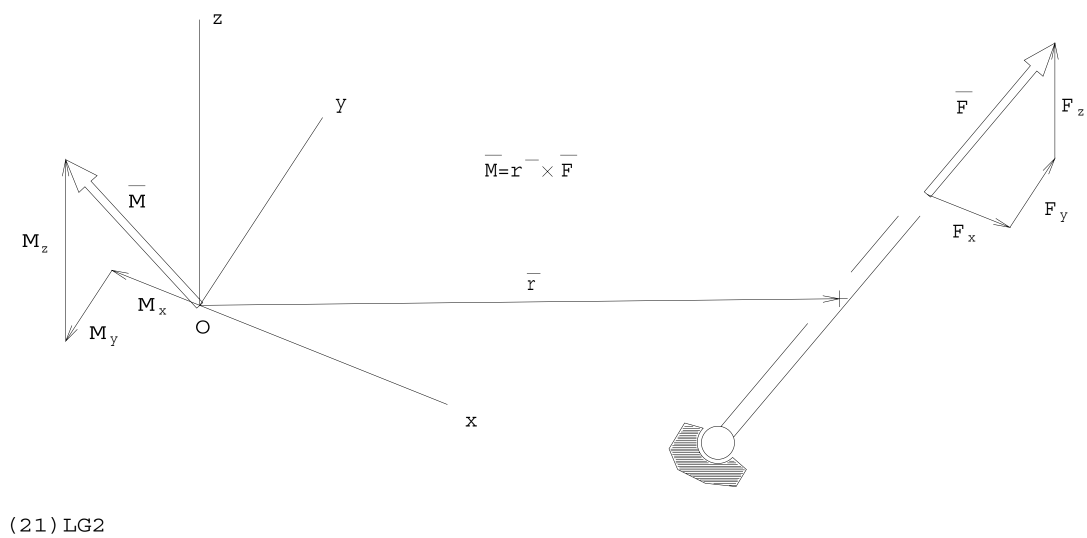

Figure 2: Force Direction and Moment

3. PLÜCKER OR LINE COORDINATES

The line of the member may be represented by these six numbers arranged as a sequence of homogeneous coordinates.

$$\mathcal{L}_r\{F_x : F_y : F_z : M_x : M_y : M_z\} \ \text{(“radial”)}$$

or

$$\mathcal{L}_a\{M_x : M_y : M_z : F_x : F_y : F_z\} \ \text{(“axial”)}$$

These are “homogeneous” because the coordinates can be all multiplied by any real number, but not zero, to represent any load magnitude but the line or member does not change its location or direction.

4. WHY RADIAL?

The line may be defined on two given points A and B. The direction numbers are given by the difference of point position vectors $\mathbf{b} - \mathbf{a}$ (or $\mathbf{a} - \mathbf{b}$ would do just as well). The homogeneous coordinates of the points are $A\{a_0 : a_1 : a_2 : a_3\}$ and $B\{b_0 : b_1 : b_2 : b_3\}$ Their ordinary Cartesian coordinates are $A(a_1/a_0, a_2/a_0, a_3/a_0)$ and $B(b_1/b_0, b_2/b_0, b_3/b_0)$. Setting $a_0 = b_0 = 1$ it is not hard to see that

$$\begin{aligned} \mathcal{L}_r\{b_1 - a_1 : b_2 - a_2 : b_3 - a_3 : \\ a_2b_3 - b_2a_3 : a_3b_1 - b_3a_1 : a_1b_2 - b_1a_2\} \end{aligned}$$

which is abbreviated as

$$\mathcal{L}_r\{p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}\}$$

5. WHY AXIAL?

The line may be defined on two given planes a and b. The plane normal determines the last three homogeneous plane coordinates. One may short-cut details by reciprocating the three principal axis intercepts, multiplying by their product and negating the first coordinate.

$$\begin{aligned} a\left\{1 : \frac{-1}{I_x} : \frac{-1}{I_y} : \frac{-1}{I_z}\right\} \\ \equiv a\{-I_xI_yI_z : I_yI_z : I_zI_x : I_xI_y\} \end{aligned}$$

which is abbreviated as

$$a\{A_0 : A_1 : A_2 : A_3\}$$

and similarly for plane b. Then the axial Plücker coordinates of the line on two planes is computed like the one on two points.

$$\begin{aligned} \mathcal{L}_a\{B_1 - A_1 : B_2 - A_2 : B_3 - A_3 : \\ A_2B_3 - B_2A_3 : A_3B_1 - B_1A_3 : A_1B_2 - B_1A_2\} \end{aligned}$$

which is abbreviated as

$$\mathcal{L}_a\{P_{01} : P_{02} : P_{03} : P_{23} : P_{31} : P_{12}\}$$

6. PLÜCKER CONDITION & KLEIN QUADRIC

Why should we need six coordinates to describe a spatial line defined by only four conditions or degrees of freedom? Imagine the x-y coordinates of a line intersecting plane $z = 0$ and y-z coordinates of its intercept on $x = 0$. Look at **2.**. The force and moments vectors must be perpendicular.

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \cdot \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = 0$$

So it is necessary that

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$$

This is the equation of a quadric hypersurface in a 5D projective space. Only points on this quadric represent lines!

7. LINE EQUATIONS

Consider the familiar parametric form

$$\mathbf{p} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$$

where $\mathbf{p}$ is the postion vector of *any* point P on the line spanning points A and B. Such points consitute a one degree of freedom set specified by the free parameter t. If $t = 0$, $\mathbf{p} = \mathbf{a}$, if $t = 1$, $\mathbf{p} = \mathbf{b}$, if $t = \frac{1}{2}$, P is midway and so on …giving nice geometric insight but no line coordinates.

8. LINEAR LINE COMPLEX

A linear equation in *line* coordinates

$$\begin{aligned} Q_{01}p_{01} + Q_{02}p_{02} + Q_{03}p_{03} \\ +Q_{23}p_{23} + Q_{31}p_{31} + Q_{12}p_{12} = 0 \end{aligned}$$

may represent one of three things.

- If lines $\mathcal{P}_r$ and $\mathcal{Q}_a$ are specified, then $\exists \mathcal{P}_r \cap \mathcal{Q}_a$. The lines intersect.
- If coordinates of $\mathcal{P}_r$ are variables the equation represents *all* lines intersecting $\mathcal{Q}_a$; a sort of "fuzz-stick" where $\mathcal{Q}_a$ is the stick.
- Things get more complicated if "line" $\mathcal{Q}_a$ doesn't satisfy the Plücker condition **6.** The component of its moment vector that together with its direction vector *do* constitute a line defines the axis of a screw whose lead is defined by the remaining moment vector component. All $\mathcal{P}_r$, the fuzz on our fuzz-stick, are perpendicular to some helical screw line.

9. LINEAR LINE COMPLEX
so what?

My copy of "Meriam & Kraige", MECHANICS (Statics) contains about three pages of illustration and explanation to explain why and how a set of six simple constraining forces and/or moments may or may not satisfy the conditions of static equilibrium. If one represents any reaction system as a set of lines, with pure couples as lines with no direction vector part, then the system is stable if the 6×6 determinant of line coordinates does not equal zero. Given six reaction lines $\mathcal{P}_r$, $\mathcal{Q}_r$, $\mathcal{R}_r$, $\mathcal{S}_r$, $\mathcal{T}_r$ and $\mathcal{U}_r$

$$\begin{vmatrix} p_{01} & p_{02} & p_{03} & p_{23} & p_{31} & p_{12} \\ q_{01} & q_{02} & q_{03} & q_{23} & q_{31} & q_{12} \\ r_{01} & r_{02} & r_{03} & r_{23} & r_{31} & r_{12} \\ s_{01} & s_{02} & s_{03} & s_{23} & s_{31} & s_{12} \\ t_{01} & t_{02} & t_{03} & t_{23} & t_{31} & t_{12} \\ u_{01} & u_{02} & u_{03} & u_{23} & u_{31} & u_{12} \end{vmatrix} \neq 0$$

Of course axial coordinates would do as well.

10. PIERCING POINT
& SPANNING PLANE

Given plane a and line $\mathcal{B}_r$, find point $P = \mathcal{B}_r \cap a$. Without going into the derivation

$$p_i = \sum_{j=0}^{3} b_{ij} A_j, \quad b_{ii} = 0, \quad b_{ji} = -b_{ij}$$

$$\begin{array}{rcccc} p_0 = & & b_{01}A_1 & +b_{02}A_2 & +b_{03}A_3 \\ p_1 = & -b_{01}A_0 & & +b_{12}A_2 & -b_{31}A_3 \\ p_2 = & -b_{02}A_0 & -b_{12}A_1 & & +b_{23}A_3 \\ p_3 = & -b_{03}A_0 & +b_{31}A_1 & -b_{23}A_2 & \end{array}$$

Given point A and line $\mathcal{B}_a$, find plane $p = \mathcal{B}_a \cap A$. Without going into the derivation

$$P_i = \sum_{j=0}^{3} B_{ij} a_j, \quad B_{ii} = 0, \quad B_{ji} = -B_{ij}$$

$$\begin{array}{rcccc} P_0 = & & B_{01}a_1 & +B_{02}a_2 & +B_{03}a_3 \\ P_1 = & -B_{01}a_0 & & +B_{12}a_2 & -B_{31}a_3 \\ P_2 = & -B_{02}a_0 & -B_{12}a_1 & & +B_{23}a_3 \\ P_3 = & -B_{03}a_0 & +B_{31}a_1 & -B_{23}a_2 & \end{array}$$

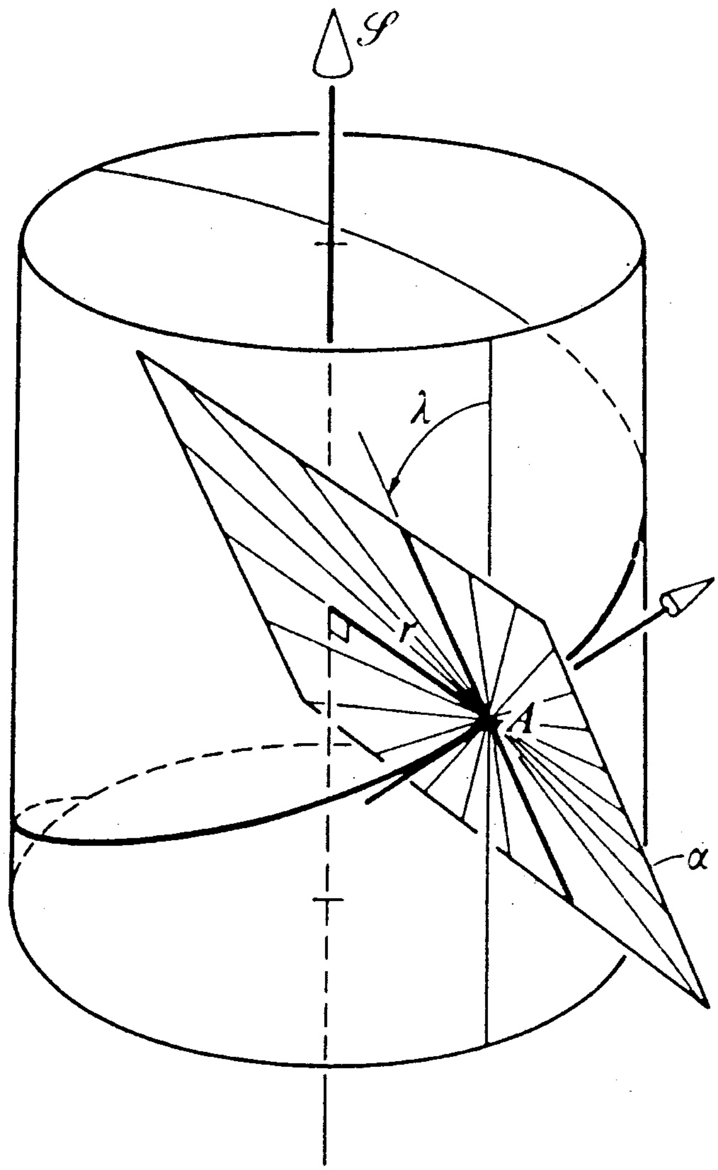

Figure 3: The Pole A of a Polar Plane α of a Linear Line Complex Showing the Helix That Passes Through A Normal to α

11. POINT ON LINE, PLANE ON LINE & LINE INTERSECTION

If all $P_i = 0$ then $A \in \mathcal{B}_a$. If all $p_i = 0$ then $a \in \mathcal{B}_r$. Note that any two equations in either set are linearly independent. Given condition **8.** for two lines $\mathcal{P}_a$ and $\mathcal{Q}_a$, find intersection point X. Choose two equations in P_{ij} from the second set in **10.** and one in Q_{ij}, say,

$$\begin{array}{rrrr} & P_{01}x_1 & +P_{02}x_2 & +P_{03}x_3 = 0 \\ -P_{01}x_0 & & +P_{12}x_2 & -P_{31}x_3 = 0 \\ -Q_{02}x_0 & -Q_{12}x_1 & & +Q_{23}x_3 = 0 \end{array}$$

and solve homogeneously for point coordinates x_i.

$$x_0 = \begin{vmatrix} P_{01} & P_{02} & P_{03} \\ 0 & P_{12} & -P_{31} \\ -Q_{12} & 0 & Q_{23} \end{vmatrix}, \quad x_1 = -\begin{vmatrix} 0 & P_{02} & P_{03} \\ -P_{01} & P_{12} & -P_{31} \\ -Q_{02} & 0 & Q_{23} \end{vmatrix}$$

$$x_2 = \begin{vmatrix} 0 & P_{01} & P_{03} \\ -P_{01} & 0 & -P_{31} \\ -Q_{02} & -Q_{12} & Q_{23} \end{vmatrix}, \quad x_3 = -\begin{vmatrix} 0 & P_{01} & P_{02} \\ -P_{01} & 0 & P_{12} \\ -Q_{02} & -Q_{12} & 0 \end{vmatrix}$$