MECH 576
Geometry in Mechanics

December 8, 2009

Instantaneous Twist Axis and Angular Velocity from Velocity of Three Points

1 Method and Axis Direction

Ab initio, one is given, on some rigid body, the position vectors of three points A, B, C as $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and their instantaneous velocity vectors $\mathbf{v}_A, \mathbf{v}_B, \mathbf{v}_C$, respectively. The following steps to obtain the instantaneous twist or *screw rate* will be used.

- Obtain the twist axis direction vector $\mathbf{t}$.
- Obtain the three point velocity component vectors $\mathbf{v}_{A\perp}, \mathbf{v}_{B\perp}, \mathbf{v}_{C\perp}$ normal to $\mathbf{t}$.
- Rotate the system of point position and velocity vectors so that $\mathbf{t}$ becomes parallel to a principal Cartesian axis, say, z.
- Project the rotated points on plane $z = 0$ as well as the rotated normal velocity vectors.
- Obtain the intersection T of lines, on these projections of A, B, C, normal to the rotated images of the normal velocity vectors. All three such lines will concur on a common point, up to the limit of numerical precision.
- Compute the magnitude of angular velocity $\omega = |\mathbf{v}|/|\mathbf{r}|$ where $\mathbf{v}$ is any point, say A, normal velocity vector projection on $z = 0$ and r is the vector from T to that image of A. The sense or sign of ω is obtained as positive if

$$\mathbf{r} \times \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ r_x v_y - r_y v_x \end{bmatrix}, \quad r_x v_y - r_y v_x > 0$$

- Reverse the rotation to obtain T in the original frame so as to locate the twist axis.

Solving the following two equations homogeneously for $\mathbf{t}$

$$\mathbf{v}_A \cdot \mathbf{t} - \mathbf{v}_B \cdot \mathbf{t} = 0, \quad \mathbf{v}_B \cdot \mathbf{t} - \mathbf{v}_C \cdot \mathbf{t} = 0$$

is expressed by the determinant expansion in Eq. 1 of the following singular matrix $[\mathbf{T}]$ on its top row minors.

$$[\mathbf{T}] = \begin{vmatrix} T_1 & T_2 & T_3 \\ v_{A1}v_{B1} - v_{B1}v_{C1} & v_{A2}v_{B2} - v_{B2}v_{C2} & v_{A3}v_{B3} - v_{B3}v_{C3} \\ v_{C1}v_{A1} - v_{A1}v_{B1} & v_{C2}v_{A2} - v_{A2}v_{B2} & v_{C3}v_{A3} - v_{A3}v_{B3} \end{vmatrix} = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = 0 \tag{1}$$

This implies

$$\mathbf{T}^\top \mathbf{t} = [T_1 \;\; T_2 \;\; T_3] \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = T_1 t_1 + T_2 t_2 + T_3 t_3 = 0 \rightarrow$$

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} (v_{A2}v_{B2} - v_{B2}v_{C2})(v_{C3}v_{A3} - v_{A3}v_{B3}) - (v_{A2}v_{B2} - v_{B2}v_{C2})(v_{A3}v_{B3} - v_{B3}v_{C3}) \\ (v_{C1}v_{A1} - v_{A1}v_{B1})(v_{A3}v_{B3} - v_{B3}v_{C3}) - (v_{A1}v_{B1} - v_{B1}v_{C1})(v_{C3}v_{A3} - v_{A3}v_{B3}) \\ (v_{A1}v_{B1} - v_{B1}v_{C1})(v_{C2}v_{A2} - v_{A2}v_{B2}) - (v_{C1}v_{A1} - v_{A1}v_{B1})(v_{A2}v_{B2} - v_{B2}v_{C2}) \end{bmatrix} \tag{2}$$

2 Unit Axis Vector and Velocity Components Normal and Parallel to It

The unit vector $\mathbf{t}_u$ of $\mathbf{t}$ is required to obtain velocity components and the rotation matrix to project the system onto a principal Cartesian plane.

$$\mathbf{t}_u = \frac{1}{\sqrt{t_1^2 + t_2^2 + t_3^2}} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \tag{3}$$

The lead or axis parallel component $\mathbf{v}_{\shortparallel}$ of the rigid body twist velocity is given by Eq. 4. Any point velocity, say, $\mathbf{v}_A \neq \mathbf{0}$ will do. Then the three $\mathbf{v}_{\perp,A,B,C}$, usually different, normal to the axis are obtained by subtraction also shown in Eq. 4.

$$\mathbf{v}_{\shortparallel} = \mathbf{v}_A \cdot \mathbf{t}_u, \;\; \mathbf{v}_{\perp A} = \mathbf{v}_A - \mathbf{v}_{\shortparallel}, \;\; \mathbf{v}_{\perp B} = \mathbf{v}_B - \mathbf{v}_{\shortparallel}, \;\; \mathbf{v}_{\perp C} = \mathbf{v}_C - \mathbf{v}_{\shortparallel} \tag{4}$$

3 Rotation Quaternion and Matrix

Choose the principal Cartesian axis, say z, that results in the largest magnitude of the product $[0 \;\; 0 \;\; 1]\mathbf{t}_u$, in that case. Find the normed rotation axis for the required projection as specified by Eq. 5.

$$\mathbf{t}_u \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_{u2} \\ -t_{u1} \\ 0 \end{bmatrix} \rightarrow \frac{1}{\sqrt{t_{u1}^2 + t_{u2}^2}} \begin{bmatrix} t_{u2} \\ -t_{u1} \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \cos\beta \\ 0 \end{bmatrix} \tag{5}$$

Find the cosine and sine of the half-angle between $\mathbf{t}_u$ and the chosen principal axis as shown in Eq. 6.

$$\frac{1}{\sqrt{t_{u1}^2 + t_{u2}^2 + (1 + t_{u3})^2}} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} t_{u1} \\ t_{u2} \\ t_{u3} \end{bmatrix} \right) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{u}$$
$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \mathbf{u} = u_3 = \cos\frac{\varphi}{2}, \;\; \sin\frac{\varphi}{2} = \left| \sqrt{1 - \cos^2\frac{\varphi}{2}} \right| \tag{6}$$

Now the normed quaternion and *Euclidian* rotation matrix are written as Eq. 7.

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \cos\frac{\varphi}{2} \\ \cos\alpha\sin\frac{\varphi}{2} \\ cos\beta\sin\frac{\varphi}{2} \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c_0^2 + c_1^2 - c_2^2 & 2c_1c_2 & 2c_0c_2 \\ 2c_2c_1 & c_0^2 - c_1^2 + c_2^2 & -2c_0c_1 \\ -2c_0c_2 & 2c_0c_1 & c_0^2 - c_1^2 - c_2^2 \end{bmatrix} = [\mathbf{R}] \tag{7}$$

4 Rotate, Project, Obtain Axis Point and Restore

Rotate A, B, C and $\mathbf{v}_{A\perp}, \mathbf{v}_{B\perp}, \mathbf{v}_{C\perp}$. The subscript $_H$ means "horizontal projection", *i.e.*, normally onto plane –in this case– $z = 0$. The rotation and projection operations in Eq. 8 pertain to point A. Similar ones take care of the other two points. Rotated normal velocity components become $\mathbf{v}_{A\perp H}, \mathbf{v}_{B\perp H}, \mathbf{v}_{C\perp H}$. These provide estimates of numerical precision because, all being normal to $\mathbf{t}$, these must have be of the form

$$\mathbf{v}_{A\perp H} = \begin{bmatrix} v_{A\perp 1} \\ v_{A\perp 2} \\ 0 \end{bmatrix}$$
$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \;\; [\mathbf{R}] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_{H1} \\ a_{H2} \\ a_{H3} \end{bmatrix} \rightarrow \begin{bmatrix} a_{H1} \\ a_{H2} \end{bmatrix}, \;\; a_{H3} \rightarrow 0 \tag{8}$$

Lines on points A_H, B_H, C_H normal, respectively, to $\mathbf{v}_{AH\perp}, \mathbf{v}_{CH\perp}, \mathbf{v}_{CH\perp}$ mutially intersect on point T_H, a point on twist axis line $\mathcal{T}$. The three lines are given by Eq. 9.

$$\begin{aligned} v_{AH\perp 1}x + v_{AH\perp 2}y - (a_{H1}v_{AH\perp 1} + a_{H2}v_{AH\perp 2}) = 0 \rightarrow X_1x + Y_1y + Z_1 = 0 \\ v_{BH\perp 1}x + v_{BH\perp 2}y - (b_{H1}v_{BH\perp 1} + b_{H2}v_{BH\perp 2}) = 0 \rightarrow X_2x + Y_2y + Z_2 = 0 \\ v_{CH\perp 1}x + v_{CH\perp 2}y - (c_{H1}v_{CH\perp 1} + c_{H2}v_{CH\perp 2}) = 0 \rightarrow X_3x + Y_3y + Z_3 = 0 \end{aligned} \tag{9}$$

Intersecting any pair of Eqs. 9 produces T_H. Intersecting all three pairs and comparing results will provide an estimate of numerical accuracy at this stage. Eq. 10 gives T_H by intersecting the first pair in Eq. 9.

$$\begin{vmatrix} T_x & T_y & T_z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} \rightarrow \mathbf{t}_H = \begin{bmatrix} t_{H1} \\ t_{H2} \end{bmatrix} = \begin{bmatrix} \frac{Y_1Z_2 - Y_2Z_1}{X_1Y_2 - X_2Y_1} \\ \\ \frac{Z_1X_2 - Z_2X_1}{X_1Y_2 - X_2Y_1} \end{bmatrix} \tag{10}$$

5 Angular Velocity and Twist Axis Line

Angular velocity magnitude $|\omega|$ can now be calculated by any one of the three expressions in Eq. 11. Differences among these, again, gives an estimate of numerical precision in computation.

$$\begin{aligned} \mathbf{r}_{AH} &= \begin{bmatrix} a_{H1} - t_{H1} \\ a_{H2} - t_{H2} \end{bmatrix}, \ |\mathbf{r}_{AH}| = \sqrt{(a_{H1} - t_{H1})^2 + (a_{H2} - t_{H2})^2}, \ |\omega| = |\mathbf{v}_{A\perp H}|/|\mathbf{r}_{AH}| \\ \mathbf{r}_{BH} &= \begin{bmatrix} b_{H1} - t_{H1} \\ b_{H2} - t_{H2} \end{bmatrix}, \ |\mathbf{r}_{BH}| = \sqrt{(a_{H1} - t_{H1})^2 + (a_{H2} - t_{H2})^2}, \ |\omega| = |\mathbf{v}_{B\perp H}|/|\mathbf{r}_{BH}| \\ \mathbf{r}_{CH} &= \begin{bmatrix} c_{H1} - t_{H1} \\ c_{H2} - t_{H2} \end{bmatrix}, \ |\mathbf{r}_{CH}| = \sqrt{(c_{H1} - t_{H1})^2 + (c_{H2} - t_{H2})^2}, \ |\omega| = |\mathbf{v}_{C\perp H}|/|\mathbf{r}_{CH}| \end{aligned} \tag{11}$$

To get the sign of ω one takes the sign of, say,

$$\mathbf{r}_{AH} \times \mathbf{v}_{A\perp H} = \begin{bmatrix} 0 \\ 0 \\ r_{AH1}v_{A\perp H2} - r_{AH2}v_{A\perp H1} \end{bmatrix} \tag{12}$$

To get point T with position vector $\mathbf{t}'$ in the original, unrotated frame the reverse rotation is invoked.

$$[\mathbf{R}]^{-1} \begin{bmatrix} t_{H1} \\ t_{H2} \\ 0 \end{bmatrix} = \mathbf{t}' = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \tag{13}$$

6 Planes on Points and Normal to Axis Normal Velocity Components

The procedure described above is deemed to be robust and immune to most problems associated with "special case" point and velocity selections. However a simpler one can be used in most general situations. This involves obtaining the twist axis line Plücker coordinates directly by intersecting any two planes, say on A and B, respectively, that are also respectively normal to $\mathbf{v}_{A\perp}$ and $\mathbf{v}_{B\perp}$. To get a point on the axis, intersect the line with any plane normal to it however to compute $|\omega|$ that plane should be on A, B or C. Assuming that $\mathbf{t}$ has been computed according to 2 and the three normal velocity components $\mathbf{v}_{\perp A}, \mathbf{v}_{\perp B}, \mathbf{v}_{\perp C}$ have been obtained with Eq. 4. Taking two, say, $\mathbf{v}_{\perp A}$ and $\mathbf{v}_{\perp B}$, form the planes

$$\begin{aligned} a\{A_0 : A_1 : A_2 : A_3\} = \{-(v_{\perp A1}a_1 + v_{\perp A2}a_3 + v_{\perp A3}a_3) : v_{\perp A1} : v_{\perp A2} : v_{\perp A3}\} \\ b\{B_0 : B_1 : B_2 : B_3\} = \{-(v_{\perp B1}b_1 + v_{\perp B2}b_3 + v_{\perp B3}b_3) : v_{\perp B1} : v_{\perp B2} : v_{\perp B3}\} \end{aligned} \tag{14}$$

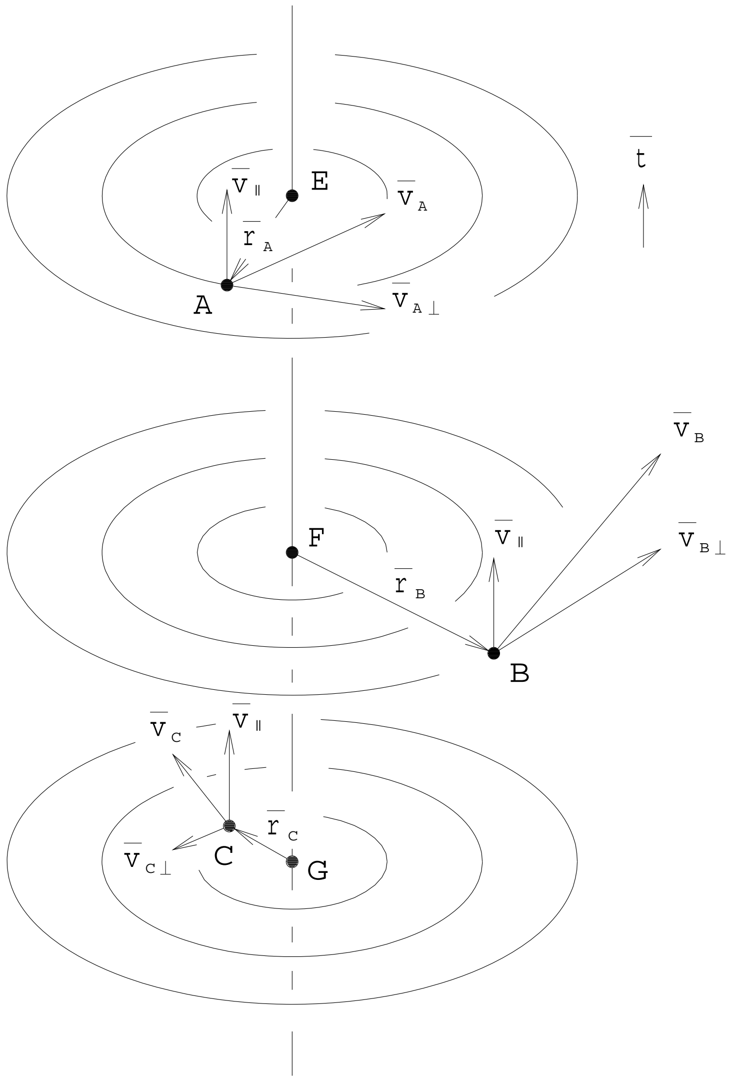

Figure 1: There Points, Three Velocities, Their Components and the Twist Axis

A picture may be helpful here so Fig. 1 is supplied. Now the axial Plücker coordinates of the twist axis line $\mathcal{T}$ are immediately available as Eq. 15.

$$\begin{aligned}\mathcal{T}_a\{T_{01}:T_{02}:T_{03}:T_{23}:T_{31}:T_{12}\} &\equiv \mathcal{T}_r\{t_{01}:t_{02}:t_{03}:t_{23}:t_{31}:t_{12}\} \\ =\{A_0B_1-A_1B_0:A_0B_2-A_2B_0:A_0B_3-A_3B_0&:A_2B_3-A_3B_2:A_3B_1-A_1B_3:A_1B_2-A_2B_1\} \\ \equiv\{a_2b_3-a_3b_2:a_3b_1-a_1b_3:a_1b_2-a_2b_1&:a_0b_1-a_1b_0:a_0b_2-a_2b_0:a_0b_3-a_3b_0\}\end{aligned} \tag{15}$$

Next, find point $E=\mathcal{T}_r\cap e$. E is the point on the axis at the foot of the perpendicular to it from A and e is the plane on A and normal to $\mathbf{t}$. Note that F,f or G,g would serve as well, using B and C, respectively. Coordinates of e are give by Eq. 16.

$$e\{E_0:E_1:E_2:E_3\}=\{-(a_1t_1+a_2t_2+a_3t_3):t_1:t_2:t_3\} \tag{16}$$

Then pierce e with $\mathcal{T}_r$ to get point E. This procedure appears below Eq. 17.

$$E = \mathcal{T}_r \cap e \equiv e_i = \sum_{j=0}^{3} t_{ij} E_j \tag{17}$$

$$\begin{array}{lllll}
e_0 = & & t_{01}E_1 & +t_{02}E_2 & +t_{03}E_3 \\
e_1 = & -t_{01}E_0 & & +t_{12}E_2 & -t_{31}E_3 \\
e_2 = & -t_{02}E_0 & -t_{12}E_1 & & +t_{23}E_3 \\
e_3 = & -t_{03}E_0 & +t_{31}E_1 & -t_{23}E_2 &
\end{array}$$

Of course the position vector of E is $\mathbf{e} = [e_1/e_0 \;\; e_2/e_0 \;\; e_3/e_0]^\top$. The radius vector from E to A is $\mathbf{r}_A = \mathbf{a} - \mathbf{e}$ and as before. Angular velocity magnitude is available as $|\omega| = \frac{|\mathbf{v}_{\perp A}|}{|\mathbf{r}_A|}$. Finally the sign of ω remains unchanged if $\mathbf{r}_A \times \mathbf{v}_{\perp A} = k\mathbf{t}$ where $k > 0$.