

MECH 576

Geometry in Mechanics

November 19, 2009

Gram-Schmidt Orthogonalization and Upper Triangularization

1 Orthogonalization

It is shown how a basis, represented by three column vectors in the 3×3 matrix \mathbf{A} , is converted to the three orthogonal vectors in \mathbf{A}^* that span the same three dimensional homogeneous vector space. The first column \mathbf{a}_{1j}^* is taken as the first principal Cartesian axis direction numbers.

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \rightarrow \mathbf{A}^* = \begin{bmatrix} a_{00}^* & a_{01}^* & a_{02}^* \\ a_{10}^* & a_{11}^* & a_{12}^* \\ a_{20}^* & a_{21}^* & a_{22}^* \end{bmatrix}$$

The procedure is outlined below.

- Choose $\mathbf{a}_{i0}^* = \mathbf{a}_{i0}$.
- Project \mathbf{a}_{i1} on \mathbf{a}_{i0}^* to obtain the component $\mathbf{a}_{i1\parallel i0} = \frac{(\mathbf{a}_{i1} \cdot \mathbf{a}_{i0})}{\mathbf{a}_{i0}^2} \mathbf{a}_{i0}$.
- Subtract to obtain $\mathbf{a}_{i1}^* = \mathbf{a}_{i1\perp i0} = \mathbf{a}_{i1} - \mathbf{a}_{i1\parallel i0}$.
- Project \mathbf{a}_{i2} on \mathbf{a}_{i0}^* to obtain the component $\mathbf{a}_{i2\parallel i0} = \frac{(\mathbf{a}_{i2} \cdot \mathbf{a}_{i0})}{\mathbf{a}_{i0}^2} \mathbf{a}_{i0}$.
- Subtract to obtain $\mathbf{a}'_{i2} = \mathbf{a}_{i2\perp i0} = \mathbf{a}_{i2} - \mathbf{a}_{i2\parallel i0}$.
- Project \mathbf{a}'_{i2} on \mathbf{a}_{i1}^* to obtain the component $\mathbf{a}_{i2\parallel i1}^* = \frac{(\mathbf{a}'_{i2} \cdot \mathbf{a}_{i1}^*)}{\mathbf{a}_{i1}^{*2}} \mathbf{a}_{i1}^*$.
- Subtract to obtain $\mathbf{a}_{i2}^* = \mathbf{a}'_{i2\perp i1} = \mathbf{a}'_{i2} - \mathbf{a}_{i2\parallel i1}^*$.

1.1 Numerical Example

To illustrate Gram-Schmidt orthogonalization consider the following example of an arbitrary 3×3 full rank numerical matrix.

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \equiv \begin{bmatrix} 5 & 9 & 6 \\ 7 & -4 & -1 \\ 2 & 3 & -8 \end{bmatrix}$$

Although any of the three column vectors could have been chosen to provide (first) principal axis direction numbers, vector $\mathbf{a}_{i0}^* = \mathbf{a}_{i0}$ was selected and \mathbf{a}_{i1} will be conditioned to provide \mathbf{a}_{i1}^* , the next.

$$\mathbf{a}_{i1\parallel i0} = \frac{1}{5^2 + 7^2 + 2^2} \left(\begin{bmatrix} 9 \\ -4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} \right) \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} = \frac{45 - 28 + 6}{78} \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} = \frac{23}{78} \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} = \frac{1}{78} \begin{bmatrix} 115 \\ 161 \\ 46 \end{bmatrix}$$

To maintain vectors in integer form the following difference can be multiplied through by 78.

$$\mathbf{a}_{i1}^* = \mathbf{a}_{i1\perp i0} = 78 \begin{bmatrix} 9 \\ -4 \\ 3 \end{bmatrix} - \begin{bmatrix} 115 \\ 161 \\ 46 \end{bmatrix} = \begin{bmatrix} 702 & -115 \\ -312 & -161 \\ 234 & -46 \end{bmatrix} = \begin{bmatrix} 587 \\ -473 \\ 188 \end{bmatrix}$$

This was a one step process because the component $\mathbf{a}_{i1}^* = \mathbf{a}_{i1\perp i0}$ was chosen to provide direction numbers of the second principal Cartesian axis perpendicular to $\mathbf{a}_{i0}^* = \mathbf{a}_{i0}$. The next and final process involves two steps because the third orthogonal basis vector is found by first subtracting the component of \mathbf{a}_{i2} that is parallel to \mathbf{a}_{i0}^* to produce $\mathbf{a}'_{i2} = \mathbf{a}_{i2\perp i0}$ and then the component of \mathbf{a}'_{i2} that is parallel to \mathbf{a}_{i1}^* is subtracted from \mathbf{a}'_{i2} to produce \mathbf{a}_{i2}^* that is normal to both \mathbf{a}_{i0}^* and \mathbf{a}_{i1}^* .

$$\mathbf{a}_{i2\parallel i0} = \frac{1}{78} \left(\begin{bmatrix} 6 \\ -1 \\ -8 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} \right) \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} = \frac{30 - 7 - 16}{78} \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} = \frac{1}{78} \begin{bmatrix} 35 \\ 49 \\ 14 \end{bmatrix}$$

$$\mathbf{a}'_{i2} = \mathbf{a}_{i2\perp i0} = 78 \begin{bmatrix} 6 \\ -1 \\ -8 \end{bmatrix} - \begin{bmatrix} 35 \\ 49 \\ 14 \end{bmatrix} = \begin{bmatrix} 468 & -35 \\ -78 & -49 \\ -624 & -14 \end{bmatrix} = \begin{bmatrix} 433 \\ -127 \\ -638 \end{bmatrix}$$

$$\mathbf{a}_{i2\parallel i1^*} = \frac{1}{587^2 + 473^2 + 188^2} \left(\begin{bmatrix} 433 \\ -127 \\ -638 \end{bmatrix} \cdot \begin{bmatrix} 587 \\ -473 \\ 188 \end{bmatrix} \right) \begin{bmatrix} 587 \\ -473 \\ 188 \end{bmatrix}$$

$$\frac{254171 + 60071 - 119944}{603642} \begin{bmatrix} 587 \\ -473 \\ 188 \end{bmatrix} = \frac{194289}{603642} \begin{bmatrix} 587 \\ -473 \\ 188 \end{bmatrix} = \frac{1}{603642} \begin{bmatrix} 114052926 \\ -91902954 \\ 36528024 \end{bmatrix}$$

$$\mathbf{a}_{i2}^* = \mathbf{a}'_{i2} - \mathbf{a}_{i2\parallel i1^*} = 603642 \begin{bmatrix} 433 \\ -127 \\ -638 \end{bmatrix} - \begin{bmatrix} 114052926 \\ -91902954 \\ 36528024 \end{bmatrix}$$

$$= \begin{bmatrix} 261376986 & -114052926 \\ -76662534 & +91902954 \\ -385123596 & -36528024 \end{bmatrix} = \begin{bmatrix} 147324060 \\ 15240420 \\ -421651620 \end{bmatrix}$$

Dividing these direction numbers by 2340 yields

$$\mathbf{a}_{i2}^* = \begin{bmatrix} 62959 \\ 6515 \\ -180193 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^* = \begin{bmatrix} 5 & 587 & 62959 \\ 7 & -473 & 6513 \\ 2 & 188 & -180193 \end{bmatrix}$$

Imposing a Euclidean norm produces the three unit column vectors in

$$\mathbf{A}_n^* = \begin{bmatrix} \frac{5}{\sqrt{78}} & \frac{587}{\sqrt{603642}} & \frac{29}{\sqrt{7739}} \\ \frac{7}{\sqrt{78}} & -\frac{473}{\sqrt{603642}} & \frac{3}{\sqrt{7739}} \\ \frac{2}{\sqrt{78}} & \frac{94\sqrt{2}}{\sqrt{301821}} & -\frac{83}{\sqrt{7739}} \end{bmatrix}$$

It is easy to show that

$$|\mathbf{A}_n^*| = 1 \quad \text{and} \quad [\mathbf{A}_n^*][\mathbf{A}_n^*]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.2 Orthogonal Matrix as Rotational Operator

Examine Fig. 1.

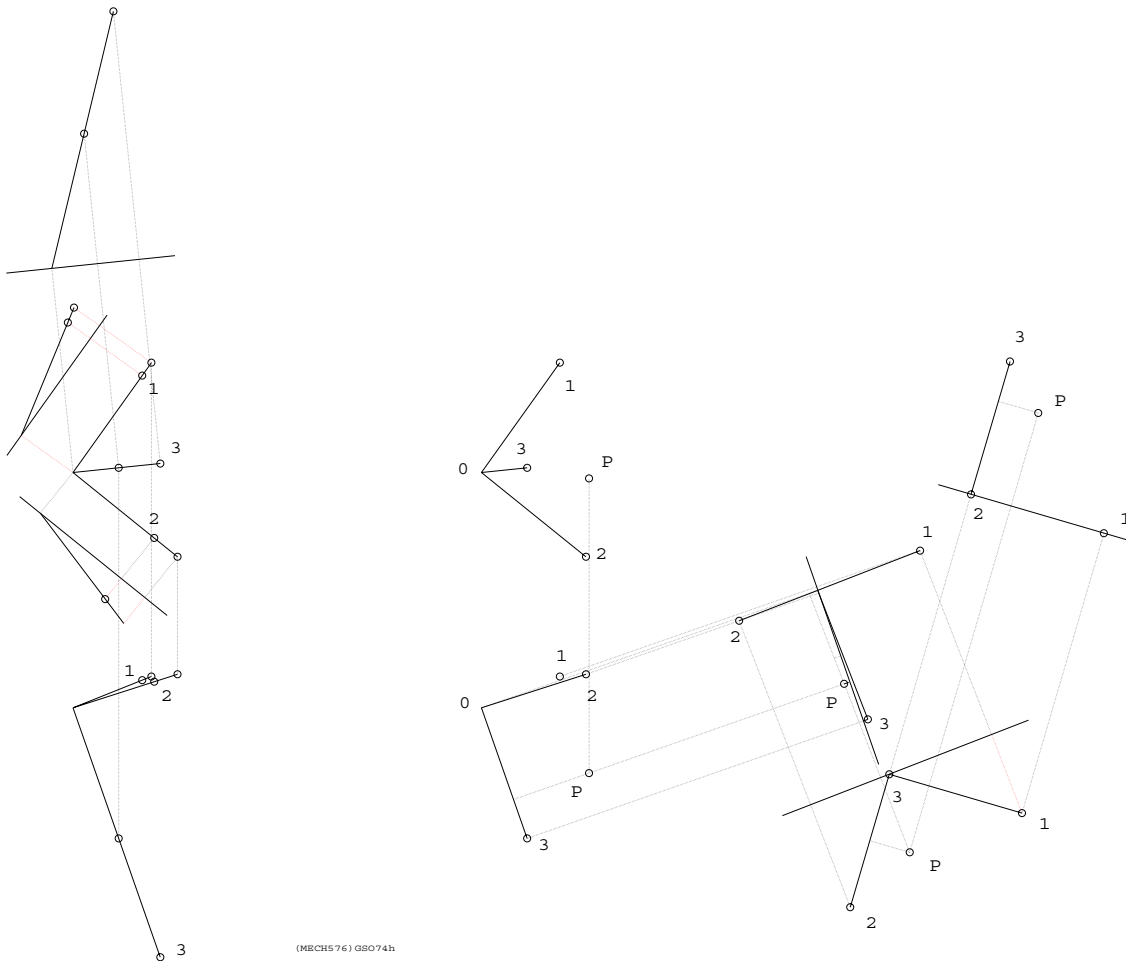


Figure 1: Representation of Matrices \mathbf{A}^* and \mathbf{A}_n^* along with Transformation of Point $P(3, 5, 7)$

At left, the orthogonal axis vector triad of the three column vectors in \mathbf{A}^* that radiates from a common origin has been plotted after dividing the second column by 100 and the third by 10000. Then a length of 10 units from the origin has been marked off on each axis. This “normalized” frame is reproduced at the centre of the figure. Finally, via three auxiliary projections, two conjugate views of the frame have been produced so that one may take the vector 1 as x -axis, 2 as y -axis and 3 as z -axis. The view at upper right is a “front” view and its partner at lower right is an “under-side” view of this right handed Cartesian frame in which the point $P(3, 5, 7)$ has been plotted. When this point is projected back to the conjugate view pair in the middle of the figure, where the three column vectors of \mathbf{A}_n^* were plotted in top and front views, one measures its coordinates as, approximately, $P(7.784, -0.427, -4.715)$, the same result is obtained with the following matrix multiplication.

$$\begin{bmatrix} \frac{5}{\sqrt{78}} & \frac{587}{\sqrt{603642}} & \frac{29}{\sqrt{7739}} \\ \frac{7}{\sqrt{78}} & -\frac{473}{\sqrt{603642}} & \frac{3}{\sqrt{7739}} \\ \frac{2}{\sqrt{78}} & \frac{94\sqrt{2}}{\sqrt{301821}} & -\frac{83}{\sqrt{7739}} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 7.783599778 \\ -0.4274821649 \\ -4.715170571 \end{bmatrix}$$

2 Upper Triangularization and Matrix Rank

The numerical matrix used in the orthogonalization example above will now be used to illustrate that a full rank square matrix will contain no rows with all zero elements, *i.e.*, the diagonal elements are all non-zero, after all elements beneath the diagonal $a_{01}, a_{12}, \dots, a_{n-1,n}$ have been zeroed by Gaussian elimination.

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} a''_{00} & a''_{01} & a''_{02} \\ 0 & a''_{11} & a''_{12} \\ 0 & 0 & a''_{22} \end{bmatrix}$$

$$\begin{bmatrix} 5 & 9 & 6 \\ 7 & -4 & -1 \\ 2 & 3 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 9 & 6 \\ 0 & -83 & -47 \\ 0 & -3 & -52 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 9 & 6 \\ 0 & -83 & -47 \\ 0 & 0 & 4175 \end{bmatrix}$$

The original 3×3 numerical example matrix was upper triangularized by the following sequence of subtractions.

$$\begin{aligned} \mathbf{a}''_{1j} &= 5\mathbf{a}_{0j} - 7\mathbf{a}_{1j} \\ \mathbf{a}^{\#}_{2j} &= 5\mathbf{a}_{2j} - 2\mathbf{a}_{0j} \\ \mathbf{a}''_{2j} &= -83\mathbf{a}^{\#}_{2j} - (-3\mathbf{a}''_{1j}) \end{aligned}$$

It is obvious that this is not an orthogonal matrix. It would be diagonal if it were. If normalized, it would be a unitary matrix.

2.1 Upper Triangularization and Rank Deficiency

Consider the matrix below.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -5 & -4 & -3 \end{bmatrix}$$

Its third row has been deliberately constructed as

$$\mathbf{a}_{2j} = 3\mathbf{a}_{0j} - 2\mathbf{a}_{1j}$$

Subtracting $\mathbf{a}_{1j} - 4\mathbf{a}_{0j}$ and $\mathbf{a}_{2j} - (-5\mathbf{a}_{0j})$ produces

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 6 & 12 \end{bmatrix}$$

It is now obvious that elements of the last row are in the same ratio as those of the second. The next elimination step produces

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

so the rank of this matrix is two.

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