MECH 576
Geometry in Mechanics

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Grassmannian Reduction of Quadratic Forms

Abstract: Conic sections, *i.e.*, ellipses, hyperbolæ and parabolæ, are encountered in elementary geometry but chopping cones with planes is regarded as mathematical entertainment, not a serious engineering activity. A mistaken view. Conics have a way of turning up in mechanics as integrals of linear forms like $m\int v dv = \frac{1}{2}mv^2$, $k\int x dx = \frac{1}{2}kx^2$, $R\int i di = \frac{1}{2}Ri^2$ and $F\int x dx = \frac{1}{2}Fx^2 = EI\frac{dy}{dx}$. Hermann Grassmann's "Extension Theory" (*Ausdenungslehre*), based on determinant expansion, will be used to reduce a conic on five given points to illustrate the computational efficiency of this method. This example is preceded by an introduction to singular matrices and planar projectivity and to Grassmannian analysis of linear forms, *i.e*, point, plane and line. Then simple quadratic problems, about circles on three points and spheres on four, are exposed. Importance of general quadratic coordinates, choosing an optimal frame and symbolic computation are discussed.

1 Linear Forms: An Introduction

1.1 The Plane Equation: A plane on Three Given Points

Given points $P_i\{w_i : x_i : y_i : z_i\}$, $i = 1, 2, 3$, the equation of plane p is expressed by expanding the following 4×4 determinant on top row minors.

$$p : \begin{vmatrix} w & x & y & z \\ w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \\ w_3 & x_3 & y_3 & z_3 \end{vmatrix} = Ww + Xx + Yy + Zz = 0 \tag{1}$$

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} w - \begin{vmatrix} w_1 & y_1 & z_1 \\ w_2 & y_2 & z_2 \\ w_3 & y_3 & z_3 \end{vmatrix} x + \begin{vmatrix} w_1 & x_1 & z_1 \\ w_2 & x_2 & z_2 \\ w_3 & x_3 & z_3 \end{vmatrix} y - \begin{vmatrix} w_1 & x_1 & y_1 \\ w_2 & x_2 & y_2 \\ w_3 & x_3 & y_3 \end{vmatrix} z = 0 \tag{2}$$

This equation states that *any* point P, with the homogeneous coordinates $\{w : x : y : z\}$, which satisfies it, is on the plane p with homogeneous *plane* coordinates $\{W : X : Y : Z\}$. Cartesian coordinates of P in Euclidean space are obtained by dividing each row of the 4×4 determinant matrix by its first element, *i.e.*, w or w_i.

1.2 The Point Equation: A Point on Three Planes

Given planes $p_i\{W_i : X_i : Y_i : Z_i\}$, $i = 1, 2, 3$, the equation of point P is expressed by expanding the following 4×4 determinant on top row minors.

$$P : \begin{vmatrix} W & X & Y & Z \\ W_1 & X_1 & Y_1 & Z_1 \\ W_2 & X_2 & Y_2 & Z_2 \\ W_3 & X_3 & Y_3 & Z_3 \end{vmatrix} = wW + xX + yY + zZ = 0 \tag{3}$$

$$\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} W - \begin{vmatrix} W_1 & Y_1 & Z_1 \\ W_2 & Y_2 & Z_2 \\ W_3 & y_3 & Z_3 \end{vmatrix} X + \begin{vmatrix} W_1 & X_1 & Z_1 \\ W_2 & X_2 & Z_2 \\ W_3 & X_3 & Z_3 \end{vmatrix} Y - \begin{vmatrix} W_1 & X_1 & Y_1 \\ W_2 & X_2 & Y_2 \\ W_3 & X_3 & Y_3 \end{vmatrix} Z = 0 \tag{4}$$

This equation states that *any* plane p, with homogeneous coordinates $\{W : X : Y : Z\}$, which satisfies it, is on the point P with homogeneous *point* coordinates $\{w : x : y : z\}$.

These concepts force one to regard the point as a dot-like object in point space but as a sheet-like object in plane space. Similarly in plane space, planes are dot-like objects and points become sheet-like. This is why the point and plane in three dimensional space are called *duals* of each other. The principle of duality has profound mathematical implication because any true statement or theorem concerning dual elements is equally valid if the rôles, *i.e.*, wherever they appear or are stated, of dual partners are interchanged. All this may seem arcane, pedantic -even unnecessary- but it is a crucial step to gain the confidence required to tackle quadratic forms.

1.3 Duality of Point and Line in the Plane

Here the steps in the previous two subsections are summarized by one; the line equation. The dually identical point equation is omitted and may be completed as an exercise.

$$g : \begin{vmatrix} w & x & y \\ w_1 & x_1 & y_1 \\ w_2 & x_2 & y_2 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} w - \begin{vmatrix} w_1 & y_1 \\ w_2 & y_2 \end{vmatrix} x + \begin{vmatrix} w_1 & x_1 \\ w_2 & x_2 \end{vmatrix} y = 0 \tag{5}$$

So $Ww + Xx + Yy = 0$ is the equation of the line g on points $P\{w_1 : x_1 : y_1\}$ and $Q\{w_2 : x_2 : y_2\}$ on the spatial plane $p_0\{0 : 0 : 0 : 1\}$, *i.e.*, $z = 0$. Consider an example where $P(2, 0)$ and $Q(0, 3)$ are given to produce $g\{W : X : Y\}$. Note that () embrace ordinary Cartesian coordinates while {} enclose homogeneous ones.

$$\begin{vmatrix} w & x & y \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{vmatrix} \Rightarrow g\{6 : -3 : -2\} \tag{6}$$

Notice that $X : Y$ constitute a vector normal to and rotated counterclockwise from the vector (directed line segment) $P \to Q$ while W is the moment of the vector about origin $O\{1 : 0 : 0\}$, *i.e.*, $O(0, 0)$.

It may be observed that *planar* line coordinates are conveniently obtained by setting $W = 1$, for lines not on O, so that $X = -I_x^{-1}, Y = I_y^{-1}$ where I_x and I_y are the respective line/axis intercepts. For lines *on* O, $W = 0$ and X and Y are the direction numbers of the segment $P \to Q$, rotated counterclockwise by an angle of $\frac{\pi}{2}$. *E.g*, if P and Q are both translated so that P becomes coincident with O so that the segment becomes $P(0, 0) \to Q(-2, 3)$ the determinant expansion produces $g_O\{0 : -3 : 2\}$.

Notice that lines on the origin are on a unique point while points at ∞ are on a unique line $g_\omega\{1:0:0\}$ that closes the projective plane. This helps us to imagine objects at infinity in point space because of their dual relationship to lines -or planes in 3D space- on the origin.

The negative reciprocal intercept and normal direction rules apply similarly to points and planes in space except that the rôle of counterclockwise parity with respect to a directed line segment is replaced by the right-hand screw circulation on the three-point (plane defining) sequence in order to define the sense of the normal. Either sense may be valid, using homogeneous coordinates but the right hand screw parity convention is preserved.

1.4 Self Duality of Lines in Space

Consider the line on two given points in space expressed by a 4×4 doubly rank deficient matrix determinant. The top row is a column index to keep track of the double subscripts that are conventionally attached to the six homogeneous Plücker coordinates that represent spatial lines.

$$\begin{vmatrix} (0) & (1) & (2) & (3) \\ w & x & y & z \\ w' & x' & y' & z' \\ w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \end{vmatrix} = 0 \tag{7}$$

This is expanded on the 2×2 minors of the top two rows containing the two variable point coordinates. Coefficients generated by this process from the elements of the two bottom rows are *ray* line or Plücker coordinates of $\mathcal{G}_r$ on points P and Q.

$$\mathcal{G}_r\{(w_1x_2 - w_2x_1) : (w_1y_2 - w_2y_1) : (w_1z_2 - w_2z_1) : (y_1z_2 - y_2z_1) : (z_1x_2 - z_2x_1) : (x_1y_2 - x_2y_1)\}$$
$$\equiv \{p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}\} \tag{8}$$

The first three coordinates are the line direction numbers while the last three are the components of the moment of these numbers about O. These are Homogeneous coordinates so $p_{01}^2 + p_{02}^2 + p_{03}^2 + p_{23}^2 + p_{31}^2 + p_{12}^2 \neq 0$ and it may be shown that $p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$.

If a pair of plane coordinates are expanded in a similar way the resulting Plücker coordinates of the $\mathcal{G}_a\{P_{01} : P_{02} : P_{03} : P_{23} : P_{31} : P_{12}\}$ are called *axial.* The first three are the moment components of the line direction numbers corresponding to the last three. If a pair of given points and a pair of given planes are on the same line then

$$\{p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}\} = c\{P_{23} : P_{31} : P_{12} : P_{01} : P_{02} : P_{03}\} \tag{9}$$

where c is any non-zero multiplier. More will be said about generation Plücker coordinates, later.

1.5 Incidence Relationships on Linear Forms

1.5.1 The Plane on a Given Point and Axial Line

$$p = \mathcal{G}_a \cap P : P_i = \sum_{j=0}^{3}(1 - \delta_{ij})p_j G_{ij} \tag{10}$$

1.5.2 The Point on a Given Plane and Radial Line

$$P = \mathcal{G}_r \cap p : p_i = \sum_{j=0}^{3} (1 - \delta_{ij}) P_j g_{ij} \tag{11}$$

1.5.3 Symbols

$$\begin{aligned} p\{P_0 : P_1 : P_2 : P_3\} \\ P\{p_0 : p_1 : p_2 : p_3\} \\ \mathcal{G}_a\{G_{01} : G_{02} : G_{03} : G_{23} : G_{31} : G_{12}\} \\ \mathcal{G}_r\{g_{01} : g_{02} : g_{03} : g_{23} : g_{31} : g_{12}\} \\ \delta_{ij} = 1,\ i = j;\quad \delta_{ij} = 0,\ i \neq j;\quad P_{ji} = -P_{ij};\quad p_{ji} = -p_{ij} \end{aligned}$$

Proof[1] of these relationships is not but too long to be included here. This will be explained later.

1.6 Reduced Linear Forms

In the case of point and plane, there are four reduced forms -ideals- of each. These include the origin, the plane at infinity, the points that close the principal axes and the principal planes.

$$\begin{aligned} P_O\{c : 0 : 0 : 0\},\ p_\infty\{c : 0 : 0 : 0\} \\ P_{\infty x}\{0 : c : 0 : 0\},\ p_{x=0}\{0 : c : 0 : 0\} \\ P_{\infty y}\{0 : 0 : c : 0\},\ p_{y=0}\{0 : 0 : c : 0\} \\ P_{\infty z}\{0 : 0 : 0 : c\},\ p_{z=0}\{0 : 0 : 0 : c\} \end{aligned}$$

Any point or plane can be reduced, respectively, by translation. As an exercise, explain why this statement is valid notwithstanding that all the points on a plane cannot be mapped to a principal plane by translation.

There are six reduced line forms, *i.e.*, three with direction numbers along respective principal coordinate axes and three on p_∞ with moments in the respective principal axial directions.

$$\begin{aligned} \mathcal{G}_{aOx}\{0 : 0 : 0 : c : 0 : 0\} \equiv \mathcal{G}_{rOx}\{c : 0 : 0 : 0 : 0 : 0\} \\ \mathcal{G}_{aOy}\{0 : 0 : 0 : 0 : c : 0\} \equiv \mathcal{G}_{rOy}\{0 : c : 0 : 0 : 0 : 0\} \\ \mathcal{G}_{aOz}\{0 : 0 : 0 : 0 : 0 : c\} \equiv \mathcal{G}_{rOz}\{0 : 0 : c : 0 : 0 : 0\} \\ \mathcal{G}_{a\infty x}\{c : 0 : 0 : 0 : 0 : 0\} \equiv \mathcal{G}_{r\infty x}\{0 : 0 : 0 : c : 0 : 0\} \\ \mathcal{G}_{a\infty y}\{0 : c : 0 : 0 : 0 : 0\} \equiv \mathcal{G}_{r\infty y}\{0 : 0 : 0 : 0 : c : 0\} \\ \mathcal{G}_{a\infty z}\{0 : 0 : c : 0 : 0 : 0\} \equiv \mathcal{G}_{r\infty z}\{0 : 0 : 0 : 0 : 0 : c\} \end{aligned}$$

Any line is reduced by rotation and translation, the same tools used to reduce quadratic forms, later.

The selection of reduced initial linear forms is the key to successful and simple symbolic solution to higher order problems.

The preceding introduction to linear forms is included to make the notion of plane and line coordinates a familiar one. Most engineers think exclusively in terms of point coordinates. This mind-set must be overcome in order to accept and skilfully use the ideas conveyed by these other, possibly higher dimensional, systems that are often convenient in dealing with spatial curves and surfaces and with kinematic mappings that represent planar, spherical and general spatial motions of a rigid body.

2 Introduction to Quadratic Forms

There are ten spatial homogeneous quadratic form variables -product combinations of two point coordinates- that may be treated like homogeneous coordinates.

$$w^2\; wx,\; wy,\; wz,\; x^2,\; xy,\; xz,\; y^2,\; yz,\; z^2$$

Treatment will be restricted to forms containing only point coordinates but bear in mind that the determinant expansions will produce combinations of embedded plane coordinates which belond to the dual, *derivative* or tangent space. This will become a little clearer when we study -conic curve and quadric surface- symmetric coefficient matrices and the notion of polarity of dual elements.

However the simple, familiar examples below should help to restore confidence that all is not hopelessly complicated.

2.1 Tetracircular Coordinates: The Circle on Three Points

This is a planar problem. This might be approached by finding the the circle centre on right-bisectors of triangle sides where the given points are vertices. Then the distance from this centre to any of the three points will define the radius. The approach described below generates the circle equation directly. Note that sum $(x^2 + y^2)$ may be regarded as a single variable due to the common coefficient on this term. Similarly, the column under xy is omitted because circle shape is invariant under rotation. Finally the homogenizing coordinate w is taken $w = 1$ because circles have no real points at infinity.

$$\begin{vmatrix} w^2 & (x^2+y^2) & wx & wy \\ 1 & (x_1^2+y_1^2) & x_1 & y_1 \\ 1 & (x_2^2+y_2^2) & x_2 & y_2 \\ 1 & (x_3^2+y_3^2) & x_3 & y_3 \end{vmatrix} = 0 \tag{12}$$

Expanding on top row element minors, as before in the case of planar line and point and spatial plane and point equations, putting the w column on the right -in reverse lexographic order as some authors use, *e.g.*, Klein[2], and choosing a computation-simplifying frame by translating given data so $x_1 = y_1 = x_2 = 0$ produces

$$\begin{vmatrix} (x^2+y^2) & wx & wy & w^2 \\ 0 & 0 & 0 & 1 \\ y_2^2 & 0 & y_2 & 1 \\ (x_3^2+y_3^2) & x_3 & y_3 & 1 \end{vmatrix} \tag{13}$$

So the circle equation becomes

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} (x^2+y^2) - \begin{vmatrix} 0 & 0 & 1 \\ y_2^2 & y_2 & 1 \\ (x_3^2+y_3^2) & y_3 & 1 \end{vmatrix} x + \begin{vmatrix} 0 & 0 & 1 \\ y_2^2 & 0 & 1 \\ (x_3^2+y_3^2) & x_3 & 1 \end{vmatrix} y - \begin{vmatrix} 0 & 0 & 0 \\ y_2^2 & 0 & y_2 \\ (x_3^2+y_3^2) & x_3 & y_3 \end{vmatrix}$$
$$= -x_3y_2(x^2+y^2) + [(x_3^2+y_3^2)y_2 - (y_2^2y_3)]x + (y_2^2x_3)y + 0 = A(x^2+y^2) + Bx + Cy = 0$$

Of course the familiar form of the circle equation and its expansion are

$$(x-x_m)^2 + (y-y_m)^2 - r^2 = 0, \quad A(x^2+y^2) + Bx + Cy + E = 0 \tag{14}$$

so that

$$2Ax_m + B = 0, \quad 2Ay_m + C = 0, \quad A(x^2+y^2) + Bx + Cy + E = 0$$
$$r = \pm\sqrt{x_m^2 + y_m^2 - \frac{E}{A}} = \pm\frac{1}{2A}\sqrt{B^2 + C^2 - 4AE}$$

Because of simplification through choice of frame $E = 0$ and $r = \sqrt{B^2 + C^2}/(2A)$ not to mention the reduction in computational effort to expand the four determinants; three, actually, since $E = 0$.

2.2 Pentaspherical Coordinates: The Sphere on Four Given Points

2.2.1 Equations

The equation of a sphere centred on $M(x_m, y_m, z_m)$, radius r, is

$$(x-x_m)^2 + (y-y_m)^2 + (z-z_m)^2 - r^2 = 0 \tag{15}$$

Expanding, the homogeneous coordinates of the sphere are seen to be $\{A : B : C : D : E\}$.

$$A(x^2+y^2+z^2) + Bx + Cy + Dz + E = 0 \tag{16}$$

Similar to the circle

$$2Ax_m + B = 0, \quad 2Ay_m + C = 0, \quad 2Az_m + D = 0, \quad A(x_m^2 + y_m^2 + z_m^2) - E = 0$$
$$r = \pm\sqrt{x_m^2 + y_m^2 + z_m^2 - \frac{E}{A}} = \pm\frac{1}{2A}\sqrt{B^2 + C^2 + D^2 - 4AE}$$

2.2.2 Expansion

Sphere coordinates may be computed with the points $P_i\{x_i : y_i : z_i : 1\},\ i = 1,2,3,4$ in the following singular determinant. Just like the plane equation defined the set of points on the plane surface with certain coefficients called coordinates, so does the sphere equation define points on the sphere specified by *its* homogeneous coordinates.

$$\begin{vmatrix} (x^2+y^2+z^2) & wx & wy & wz & w^2 \\ (x_1^2+y_1^2+z_1^2) & x_1 & y_1 & z_1 & 1 \\ (x_2^2+y_2^2+z_2^2) & x_2 & y_2 & z_2 & 1 \\ (x_3^2+y_3^2+z_3^2) & x_3 & y_3 & z_3 & 1 \\ (x_4^2+y_4^2+z_4^2) & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0 \tag{17}$$

It is suggested here that the greatest computational simplification is achieved by imposing a translation that produces $x_1 = y_1 = z_1 = x_2 = y_2 = x_3 = 0$.

3 Reduction of Quadratic Forms

Details concerning reduction of a general conic, adapted from[3], appear in section **4**. However the formulation of the three dimensional problem, *i.e.*, reducing the general quadric surface -ellipsoid, hyperboloid of one or two sheets, *etc.*- to standard form,

$$Ax^2 + A'y^2 + A''x^2 + E = 0, \tag{18}$$

given nine points on the surface, is outlined below. This requires cunning because the general Grassmannian determinant is a 10×10 array.

$$\begin{vmatrix} x^2 & y^2 & z^2 & xy & yz & zx & wx & wy & wz & w^2 \\ x_1^2 & y_1^2 & z_1^2 & x_1y_1 & y_1z_1 & z_1x_1 & w_1x_1 & w_1y_1 & w_1z_1 & w_1^2 \\ x_2^2 & y_2^2 & z_2^2 & x_2y_2 & y_2z_2 & z_2x_2 & w_2x_2 & w_2y_2 & w_2z_2 & w_2^2 \\ x_3^2 & y_3^2 & z_3^2 & x_3y_3 & y_3z_3 & z_3x_3 & w_3x_3 & w_3y_3 & w_3z_3 & w_3^2 \\ x_4^2 & y_4^2 & z_4^2 & x_4y_4 & y_4z_4 & z_4x_4 & w_4x_4 & w_4y_4 & w_4z_4 & w_4^2 \\ x_5^2 & y_5^2 & z_5^2 & x_5y_5 & y_5z_5 & z_5x_5 & w_5x_5 & w_5y_5 & w_5z_5 & w_5^2 \\ x_6^2 & y_6^2 & z_6^2 & x_6y_6 & y_6z_6 & z_6x_6 & w_6x_6 & w_6y_6 & w_6z_6 & w_6^2 \\ x_7^2 & y_7^2 & z_7^2 & x_7y_7 & y_7z_7 & z_7x_7 & w_7x_7 & w_7y_7 & w_7z_7 & w_7^2 \\ x_8^2 & y_8^2 & z_8^2 & x_8y_8 & y_8z_8 & z_8x_8 & w_8x_8 & w_8y_8 & w_8z_8 & w_8^2 \\ x_9^2 & y_9^2 & z_9^2 & x_9y_9 & y_9z_9 & z_9x_9 & w_9x_9 & w_9y_9 & w_9z_9 & w_9^2 \end{vmatrix} = 0 \tag{19}$$

Even a frame simplification will require the expansion of large minors.

$$\begin{vmatrix} x^2 & y^2 & z^2 & xy & yz & zx & wx & wy & wz & w^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & z_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ x_3^2 & y_3^2 & z_3^2 & 0 & y_3z_3 & 0 & 0 & w_3y_3 & w_3z_3 & 1 \\ x_4^2 & y_4^2 & z_4^2 & x_4y_4 & y_4z_4 & z_4x_4 & w_4x_4 & w_4y_4 & w_4z_4 & 1 \\ x_5^2 & y_5^2 & z_5^2 & x_5y_5 & y_5z_5 & z_5x_5 & w_5x_5 & w_5y_5 & w_5z_5 & 1 \\ x_6^2 & y_6^2 & z_6^2 & x_6y_6 & y_6z_6 & z_6x_6 & w_6x_6 & w_6y_6 & w_6z_6 & 1 \\ x_7^2 & y_7^2 & z_7^2 & x_7y_7 & y_7z_7 & z_7x_7 & w_7x_7 & w_7y_7 & w_7z_7 & 1 \\ x_8^2 & y_8^2 & z_8^2 & x_8y_8 & y_8z_8 & z_8x_8 & w_8x_8 & w_8y_8 & w_8z_8 & 1 \\ x_9^2 & y_9^2 & z_9^2 & x_9y_9 & y_9z_9 & z_9x_9 & w_9x_9 & w_9y_9 & w_9z_9 & 1 \end{vmatrix} = 0 \tag{20}$$

4 Reducing the Conic

One may efficiently obtain principal axis directions and centre coordinates of a conic on five given points by expanding three sub-determinants derived from the singular matrix of the conic equation coefficients. Computation entails solution od a quadratic equation in $\cos^2\phi$, where ϕ is the principal axis rotation angle with respect to the frame of the original points, and linear ones in s_0 and t_0 where these are the respective translations to centre the origin of the new, aligned frame of the conic. This closed form solution is coded in an algorithm that needs only 122 arithmetic operations.

Analysis of quadratic forms plays an important rôle in engineering. *E.g.*, one may wish to design an elliptical trammel to guide a one degree of freedom, planar manipulator end effector smoothly through five desired points. However it is not the intention here to dwell on applications and the reader may find a brief discourse in[4]. Suffice it to say that linear algebra courses, *e.g.*, Anton[5], deal with this subject and treat the reduction of the two-variable equation

$$ax^2 + b^2 + cxy + dx + ey + f = 0 \tag{21}$$

to standard form

$$a'x^2 + b'y^2 + f' = 0 \tag{22}$$

Without elaboration, the usual approach follows these steps.

- Orthogonal matrix diagonalization, determination of eigenvalues and eigenvectors and application of the two-dimensional *Principal Axis Theorem* yield the conic axis direction by eliminating the coefficient c in Eq. 21.
- Elimination of coefficients d and e in Eq. 21 produces the two required translations.

Note that Anton[5] starts with Eq. 21 and performs the diagonalization on a *numerical* example to obtain Eq. 22. On the other hand we begin with five given points to arrive at essentially the same result without ever bothering with the coefficients of Eq. 21. Furthermore the reader will see immediately that a', b', f' can be computed, from the results obtained, via three 2 determinants.

4.1 General Quadratic Form in the Plane

English language literature on the contributions of Hermann Grassmann is scarce but it is suspected that the dawning of interest in symbolic computation stimulated a complete translation of his work Kannenberg[6]. Furthermore the theory of determinants was applied to conics by Askwith[7]. In this regard the complete 6 × 6 Grassmannian determinant for the general conic is written below.

$$\begin{vmatrix} x^2 & y^2 & xy & wx & wy & w^2 \\ x_1^2 & y_1^2 & xy & x_1 & y_1 & 1 \\ x_1^2 & y_1^2 & xy & x_1 & y_1 & 1 \\ x_1^2 & y_1^2 & xy & x_1 & y_1 & 1 \\ x_1^2 & y_1^2 & xy & x_1 & y_1 & 1 \\ x_1^2 & y_1^2 & xy & x_1 & y_1 & 1 \end{vmatrix} = 0 \tag{23}$$

Extracting the symbolic coefficients requires the evaluation of six 5 determinants and produces expressions containing very many terms. Such a futile exercise will not be attempted.

4.2 Breaking Down the Problem

4.2.1 Rotation

Principal axis rotation angle ϕ is obtained by expanding only the minor on the row and column containing xy. Its determinant must vanish when coordinate axes are made parallel to conic axes of symmetry. Computation effort is reduced, without loss of generality, by placing the five given points in a reduced frame. *i.e.*, with $x_1 = y_1 = x_2 = 0$. Thus the 5 × 5 determinant of the minor on xy becomes

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ a_{23}^2 & a_{24}^2 & a_{23} & a_{24} & 1 \\ a_{i1} & a_{i2} & a_{i3} & a_{i4} & 1 \end{vmatrix} xy = 0 \tag{24}$$

such that $a_{23} = y_2 \sin\phi$, $a_{24} = y_2 \cos\phi$, $a_{i1} = a_{i3}^2$, $a_{i2} = a_{i4}^2$, $a_{i3} = x_i \cos\phi + y_i \sin\phi$, and $a_{i4} = y_i \cos\phi - x_i \sin\phi$, $i = 3, 4, 5$. Symbolic computation and some careful simplification yields a quadratic equation in $\cos^2\phi$.

$$A\cos^4\phi + B\cos^2\phi + C = 0 \tag{25}$$

The coefficients A, B, C are not too daunting, as may be seen in section **4.3.1**.

4.2.2 Translations

After performing the rotation that aligns the conic the five given points are transformed to the rotated frame such that coordinate transformation $P_i(x_i, y_i) \Rightarrow P_i(s_i, t_i)$ takes place as indicated by ϕ. The five points in the new (s, t) frame satisfy Eq. 24 so only four of them are needed to get s_0 along s and t_0 along t. These are the respective translations needed to out the conic centre on the origin. This is much simpler than the computation of ϕ with Eq. 24. It is represented by the two determinants below.

$$\begin{vmatrix} s_0^2 & 0 & 0 & 1 \\ (s_2-s_0)^2 & t_2^2 & t_2 & 1 \\ (s_3-s_0)^2 & t_3^2 & t_3 & 1 \\ (s_4-s_0)^2 & t_4^2 & t_4 & 1 \end{vmatrix} s=0, \quad \begin{vmatrix} 0 & t_0^2 & 0 & 1 \\ s_2^2 & (t_2-t_0)^2 & s_2 & 1 \\ s_3^2 & (t_3-t_0)^2 & s_3 & 1 \\ s_4^2 & (t_4-t_0)^2 & s_4 & 1 \end{vmatrix} \tag{26}$$

4.3 Algorithm

It is hoped that the reader will forgive, in the interests of brevity and a working program example, a coded listing . The seven non-trivial given point coordinates are supplied to produce $\cos\phi$, $\sin\phi$, s_0 and t_0. `I` is the only "tweak". It is required because the sign combination of $\cos\phi$ and $\sin\phi$ may wrongly selected, faced with the ambiguity $\sin\phi = \pm\sqrt{1-\cos^2\phi}$. `I`$= -1$ will correct an unsuccessful try with `I`$= 1$. The test example is illustrated in Fig. 1.

4.3.1 Programmed Example

```
100 INPUT Y2,X3,Y3,X4,Y4,X5,Y5,I:Y23=Y2-Y3:Y24=Y2-Y4:Y25=Y2-Y5:REM End of setup
110 Q1=X3*X4*(X4*(Y5-Y3)+X5*(Y3-Y4)+X3*(Y4-Y5)):Q2=X5*(Q1+Y3*Y4*(X3*Y24-X4*Y23))
120 Q3=Y5*(X4*Y3*(X5*Y23-X3*Y25)+X3*Y4*(X4*Y25-X5*Y24)):Q=2*Y2*(Q2+Q3)
130 R=Y2*(X4*X5*Y3*Y23*(X4-X5)+X3*(X5*Y4*Y24*(X5-X3)+X4*Y25*(X3-X4)))
140 P=-2*R:QQ=Q*Q:A=P*P+QQ:B=QQ-2*P*R:C=R*RA2=2*A:IF A=0 THEN STOP
150 D=B*B-4*A*C:IF D=O THEN STOP
160 DR=SQR(D):CP=SQR((B+DR)/A2):SP=I*SQR(1-CP*CP):REM End of rotation routine
170 S2=Y2*SP:T2=Y2*CP:S3=X3*CP+Y3*SP:S4=X4*CP+Y4*SP:T3=Y3*CP-X3*SP:T4=Y4*CP-X4*SP
180 REM End of rotated coord. transformation
190 S01=S2*T3*T4*(T4-T3):S02=S3*T2*T4*(T2-T4)S03=S4*T2*T3*(T3-T2)
200 DS0=2*(S01+S02+S03):IF DS0=0 THEN STOP
210 NS0=S2*S01+S3*S02+S4*S03:S0=NS0/DS0:REM End of s-axial displacement routine
220 T01=S2*S4*T3*(S4-S2):T02=S2*S3*T4*(S2-S3):T03=S3*S4*T2*(S3-S4)
230 DTO=2*(T01+T02+T03):IF DTO=0 THEN STOP
240 NT0=T3*T01+T4*T01+T2*T03:T0=NT0/DTO:REM End of s-axial displacement routine
250 PRINT CP,SP,S0,T0:STOP:END

RUN
? 7.209,9.391,8.05,-4.919,0.894,7.155,3.578,1
0.843964 0.4472752 4.000085 2.999937
BREAK IN 250
```

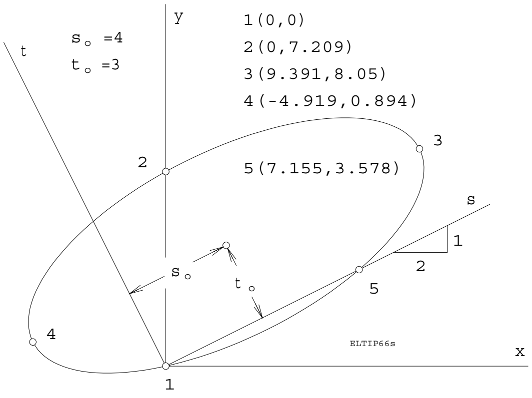

Figure 1: Reduction of an Ellipse

5 Conclusion

All theory presented above is classical geometry; largely unknown or forgotten. *Geometric thinking*, a hallmark of engineering design, includes choosing initial conditions when formulating a problem, in this case a good coordinate frame. Anton[5] mentions advances methods to do rotation - Lagrange's and Kronecker's reductions, but there is no word about Grassmann. As methods and tools for symbolic computation -still quite limited- evolve, the design engineer will become more a designer of algorithms rather than a designer of specific solutions. The latter task will fall to engineering technicians skilled in the *use* of advanced software. Similarly, advances in symbolic software will inevitably resurrect the dormant methods of Grassmann and his successors whose results were eclipsed by the "new age" of quantum and relativistic mechanics. These results have lain fallow awaiting application in a future with appropriate computational tools. One may recall that numerical analysis remained an impractical mathematical curiosity until automatic numerical computation established its importance in the '60s and '70s. We toilers in classical mechanics in the new millennium can look forward with confidence that new tools will enhance -not eliminate- our jobs, whether in practice or academe.

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