MECH 576
Geometry in Mechanics
Camera Orientation on a 3-2-1 Platform Manipulator

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1 Spatial 6dof 3-2-1 Platform Manipulator Kinematics

Examine Fig. 1. Shown there is the layout of a six legged cable driven manipulator developed at the Automation Institute at the University of Leoben by Gerhard Rath and Gerold Probst to stably move a ciné camera to record the activities of insects in the field. This is a 6dof device that is capable, within limits, of full triaxial rotational movement of the end effector (EE). Before addressing the problem at hand, *i.e.*, that of efficiently and conveniently measuring orientation of any arbitrary EE pose and achieving a specified EE orientation pose, respectively, the direct and inverse orientational kinematics of the device, the general direct and inverse positioning problems will be reviewed.

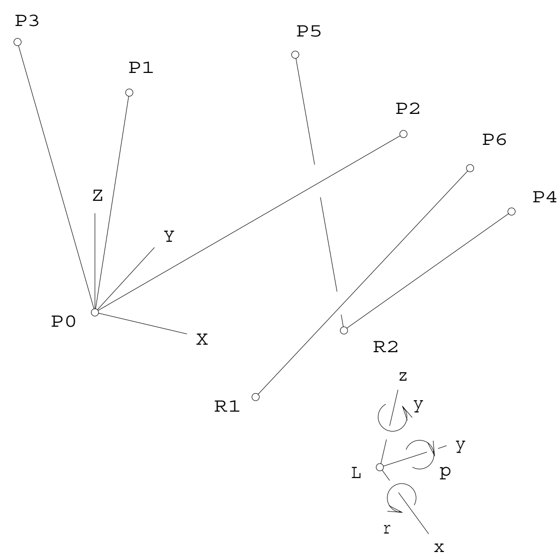

Figure 1: Two Reference Frames and Cable Anchor Points on EE and FF

The six base or fixed frame (FF) anchor points are labeled $P1$ through $P6$. Strings extend from $P1, 2, 3$ to $P0$ on EE. Two strings originating respectively at $P4$ and $P5$ on FF are attached to $R2$ on EE. The last string spans between $P6$ and $R1$. A Cartesian frame of fixed orientation is considered to be attached with its origin on $P0$ in some reference or "home" pose. Since there is a fixed displacement between the camera frame $L(x, y, z)$ and a frame natural to the three EE attachment points $P0, R1, R2$ it will be assumed that knowing the positions of these three points is equivalent to knowing the camera pose.

1.1 Inverse Displacement (IKP)

Given $P0, R1, R2$ FF coordinates, EE can be positioned by simply computing the six distances $Pi \to P0$, $i = 1, \ldots, 6$ These distances can be verified by monitoring the encoders that log the free cable lengths as string is paid out or reeled in by the six actuators.

1.2 Direct Displacement (DKP)

In this case the six string lengths are known or given and the positions of the six anchors points on EE are to be determined. This is a little more complicated than IKP but is nevertheless straight forward. The procedure is as follows.

- Find $P0$: Three spheres are imagined to be centred on $Pi({}_ip_1,\ {}_ip_2,\ {}_ip_3)$, $i = 1, 2, 3$, with respective radii r_i, the string lengths. The coordinates $P0({}_0p_1,\ {}_0p_2,\ {}_0p_2)$ are the three unknowns. Two differences between pairs of sphere equations like

  $$({}_0p_1 -_i p_1)^2 + ({}_0p_2 -_i p_2)^2 + ({}_0p_3 -_i p_3)^2 - r_i^2 = 0$$

  establish planes like

  $$2[({}_2p_1 -_1 p_1)_0p_1 + ({}_2p_2 -_1 p_2)_0p_2 + ({}_2p_3 -_1 p_3)_0p_3] +_1 p_1^2 +_1 p_2^2 +_1 p_3^2 - r_1^2 -_2 p_1^2 -_2 p_2^2 -_2 p_3^2 + r_2^2 = 0$$

  Solving these two linear equations simultaneously with any one of the original sphere equations gives two possible locations for $P0$. The one with the greater elevation, *i.e.*, value of ${}_0p_3$, will inevitably imply prohibited compressive loading on some of these three strings therefore the solution with lesser value of ${}_0p_3$ is the correct choice. The only ambiguity that can arise is in the unlikely case when the two values of ${}_0p_3$ are identical. The three sphere intersection procedure to find $P0$ will be repeated, as well as the solution choice criterion, to find $R2$ and $R1$. The three sphere intersection problem is dealt with in detail in [1]. To resolve the eight assembly mode DKP solution and to improve string length measurement accuracy one may consider adding strings, whose lengths are measured, so as to provide a 4-3-2 architecture and three four-sphere intersection problems. Four-sphere intersection is fully described in [2] and [3] and useful background material is contained in [4] and [5].

- Find $R2$: This time the three spheres are centred on $P0$, $P4$ and $P5$, with respective string lengths $P0 \to R2 = r_7$, $P5 \to R2 = r_5$ and $P6 \to R2 = r_6$.

- Find $R1$: In this final step the sphere centres are $P0$, $R2$ and $P6$ and the respective radii are $P0 \to R1 = r_8$, $R2 \to R1 = r_8$ and $P6 \to R1 = r_6$.

- Notice that r_7 and r_8 are available directly from the design parameters, *i.e.*, placement of $R2$ and $R1$ with respect to $P0$ defines these radial distances.

2 Orientation of EE Hence Camera

Assume that the coordinates of $P0, R2, R1$ are known, either via IKP or DKP, in some pose wherein one wishes to establish the orientation of EE. Call the vectors $P0 \to R1 = \mathbf{u}_1$, $P0 \to R2 = \mathbf{v}_1$

and $P0 \rightarrow L = \mathbf{w}_1$. Note that point L is the optical centre of the camera lens. Assume, also that vectors $P0 \rightarrow R1 = \mathbf{u}_0$, $P0 \rightarrow R2 = \mathbf{v}_0$ and $P0 \rightarrow L = \mathbf{w}_0$, where the camera frame is in "home" orientation, are known as well. Now the following difference vectors are formed as shown in Fig. 2.

$$\mathbf{f} = \mathbf{u}_1 - \mathbf{u}_0, \quad \mathbf{g} = \mathbf{v}_1 - \mathbf{v}_0, \quad \mathbf{h} = \mathbf{w}_1 - \mathbf{w}_0$$

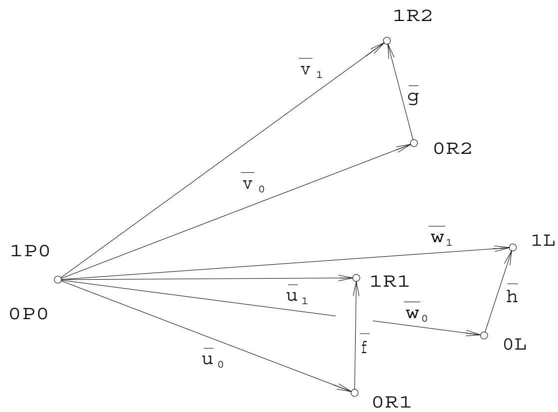

Figure 2: Point Difference Vectors Relative to $P0$

A unit vector in the direction of the axis $\mathbf{a}$, on $P0$, of a single rotation to effect this change in orientation is generally given by

$$\mathbf{a} = \frac{\mathbf{f} \times \mathbf{g}}{|\mathbf{f} \times \mathbf{g}|}$$

but if $\mathbf{g} = \mathbf{0}$ or $\mathbf{f} \parallel \mathbf{g}$ then

$$\mathbf{a} = \frac{\mathbf{f} \times \mathbf{h}}{|\mathbf{f} \times \mathbf{h}|}$$

but if $\mathbf{f} = \mathbf{0}$ or $\mathbf{f} \parallel \mathbf{h}$ then

$$\mathbf{a} = \frac{\mathbf{g} \times \mathbf{h}}{|\mathbf{g} \times \mathbf{h}|}$$

If all three choices fail then no finite rotation has taken place. Thus $\mathbf{a}$ is the direction cosine vector of the quaternion that defines this single rotation about the axis on $P0$.

$$\mathbf{Q} = \begin{bmatrix} \cos\frac{\phi}{2} \\ \cos\alpha\sin\frac{\phi}{2} \\ \cos\beta\sin\frac{\phi}{2} \\ \cos\gamma\sin\frac{\phi}{2} \end{bmatrix} \equiv \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

The rotation half angle sines and cosine, $\sin\frac{\phi}{2}$ and $\cos\frac{\phi}{2}$ are defined in terms of $t = \tan\frac{\phi}{2}$ as follows.

$$t = \frac{|\mathbf{f}|}{|\mathbf{u}|}, \quad \sin\frac{\phi}{2} = \frac{2t}{1+t^2}[\text{signum}(\mathbf{u}_0 \times \mathbf{f}) \cdot \mathbf{a}], \quad \cos\frac{\phi}{2} = \frac{1-t^2}{1+t^2}$$

Rotation matrix $\mathbf{Q}$ can now be written in terms of c_i, $i = 0, 1, 2, 3$.

$$\mathbf{Q} = \begin{bmatrix} c_0^2 + c_1^2 + c_2^2 + c_3^2 & 0 & 0 & 0 \\ 0 & c_0^2 + c_1^2 - c_2^2 - c_3^2 & 2(c_1c_2 - c_0c_3) & 2(c_1c_3 + c_0c_2) \\ 0 & 2(c_2c_1 + c_0c_3) & c_0^2 - c_1^2 + c_2^2 - c_3^2 & 2(c_2c_3 - c_0c_1) \\ 0 & 2(c_3c_1 - c_0c_2) & 2(c_3c_2 + c_0c_1) & c_0^2 - c_1^2 - c_2^2 + c_3^2 \end{bmatrix}$$

2.1 Euler Angle Decomposition

If it is desired to represent the orientation in terms of one of the many Euler angle conventions, say, a sequence of "pitch"→ "roll"→"yaw", as illustrated in Fig. 1 with turns p,r,y, the individual angles can be deduced by evaluating the angle magnitudes in the product sequence

$$[\mathbf{Y}][\mathbf{R}][\mathbf{P}] = [\mathbf{Q}]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{0y}^2 - c_{3y}^2 & -2c_{0y}c_{3y} & 0 \\ 0 & 2c_{0y}c_{3y} & c_{0y}^2 - c_{3y}^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \\ 0 & 0 & c_{0r}^2 - c_{1r}^2 & -2c_{0r}c_{1r} \\ 0 & 0 & 2c_{0r}c_{1r} & c_{0r}^2 - c_{1r}^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{0p}^2 - c_{2p}^2 & 0 & 2c_{0p}c_{2p} \\ 0 & 0 & 1 & 0 \\ 0 & -2c_{0p}c_{2p} & 0 & c_{0p}^2 - c_{2p}^2 \end{bmatrix}$$

This triple product is written as $\mathbf{C}$ below.

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & (c_{0p}^2 - c_{2p}^2)(c_{0y}^2 - c_{3y}^2) - 8c_{0p}c_{2p}c_{0r}c_{1r}c_{0y}c_{3y} \\ 0 & 2(c_{0p}^2 - c_{2p}^2)c_{0y}c_{3y} + 4c_{0p}c_{2p}c_{0r}c_{1r}(c_{0y}^2 - c_{3y}^2) \\ 0 & -2c_{0p}c_{2p}(c_{0r}^2 - c_{1r}^2) \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ -2(c_{0r}^2 - c_{1r}^2)c_{0y}c_{3y} & 2c_{0p}c_{2p}(c_{0y}^2 - c_{3y}^2) + 4(c_{0p}^2 - c_{2p}^2)c_{0r}c_{1r}c_{0y}c_{3y} \\ (c_{0r}^2 - c_{1r}^2)(c_{0y}^2 - c_{3y}^2) & 4c_{0p}c_{2p}c_{0y}c_{3y} - 2(c_{0p}^2 - c_{2p}^2)c_{0r}c_{1r}(c_{0y}^2 - c_{3y}^2) \\ 2c_{0r}c_{1r} & (c_{0p}^2 - c_{2p}^2)(c_{0r}^2 - c_{1r}^2) \end{bmatrix}$$

Noticing that all elements of $\mathbf{Q}$ are available one may proceed to evaluate the half-angle sines and cosines of the Euler angles in the context of this particular application sequence, *i.e.*, subscripts p, r, y stand for pitch, roll and yaw. Since α, β, γ are all either 0 or $\frac{\pi}{2}$ for the Euler angles about fixed principal Cartesian directions one obtains the six required values in the following evaluation sequence using corresponding elements from equations for $\mathbf{C}$ and $\mathbf{Q}$.

$$C_{32} = Q_{32}: \quad 2c_{0r}c_{1r} = 2(c_3c_2 + c_0c_1), \quad c_{0r}^2 + c_{1r}^2 = 1 \rightarrow c_{0r}, c_{1r}$$

$$C_{31} = Q_{31}: \quad -2c_{0p}c_{2p}(c_{0r}^2 - c_{1r}^2) = 2(c_3c_1 - c_0c_2), \quad c_{0p}^2 + c_{2p}^2 = 1 \rightarrow c_{0p}, c_{2p}$$

$$C_{12} = Q_{12}: \quad -2(c_{0r}^2 - c_{1r}^2)c_{0y}c_{3y} = 2(c_1c_2 - c_0c_3), \quad c_{0y}^2 + c_{3y}^2 = 1 \rightarrow c_{0y}, c_{3y}$$

There is no ambiguity in sign because one may assign a negative magnitude to the half-angle sine. Cosine of a negative half-angle is positive since the angle range may always be taken as 0 to $\pm\frac{\pi}{2}$.

References

[1] Zsombor-Murray, P.J. (2009) "An Improved Approach to the Kinematics of Clavel's Delta Robot", <http://www.cim.mcgill.ca/∼paul/Delta9Af.pdf>, 6pp.

[2] Zsombor-Murray, P.J. (2010) "Equidistant Points from a Given Pair of Spheres", <http://www.cim.mcgill.ca/∼paul/Stachel01y.pdf>, 3pp.

[3] Stachel, H. (1996) "Why Shall We also Teach the Theory behind Engineering Graphics", *Technical Report No. 35*, TU-Wien, Institute for Geometry, 5pp.

[4] Zsombor-Murray, P.J. and Linder, K. (1988) "A Descriptive Geometric Approach to Circle Construction", *Proceedings of International Conference on Engineering Graphics and Descriptive Geometry*, TU-Wien, 88-07, v.2, pp.342-349.

[5] Krames, J.L. (1929) *Vorlesungen über darstellende Geometrie von Dr. Emil Müller, II Band, Die Zyklographie*, Franz Deuticke, Leipzig u. Wien.

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