MECH 289
Design Graphics
Elementary Descriptive Geometry and Equivalent Linear Algebra

February 1, 2007

1 Introduction

Presented here, in summary, are 20 of the 23 topics listed in section **2.1** on pp.2-3 of the document **Fundamentals of Geometric Construction**. The idea is to put all the necessary linear algebra together in one place and refer to the location of the various relevant illustrations and explanations of descriptive geometry topics in that more extensive text.

2 Coordinate system ...

Refer to Fig. 1 on p.5, section **2.4**. The point label is changed from A, there, to P, here. You will have to get used to that sort of thing in the real world. However it was not done deliberately, here. The position vector of a point P is given by

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Position vector of its *top*, -H- (plan) view is

$$\mathbf{p}_H = \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix}$$

Position vector of its *front*, -F- (elevation) view is

$$\mathbf{p}_F = \begin{bmatrix} p_x \\ 0 \\ p_z \end{bmatrix}$$

Position vector of either *side*, -P- (profile) view is

$$\mathbf{p}_P = \begin{bmatrix} 0 \\ p_y \\ p_z \end{bmatrix}$$

Note that top and front views adhere to $p_z = 0$ and $p_y = 0$ but profile projection planes are customarily set at some given value of $p_x \neq 0$. However if the planes $z = 0, \quad y = 0, \quad x = 0$ are used their lines of intersection form the familiar triad of right handed Cartesian coordinate axis. If P, representing any collection of points, is projected on the plane n whose homogeneous coordinates are given by

$$n\{N_0 : N_1 : N_2 : N_3\}$$

where the four elements are just the plane coefficients, $1 = x,\ \ 2 = y,\ \ 3 = z$. Any such projection is called an *auxiliary view*. Let us call $\mathbf{p}_n$ the position vector of the image of P on n. The plane equation is written below along with the plane normal vector $\mathbf{n}$.

$$N_0 + N_1 p_{nx} + N_2 p_{ny} + N_3 p_{nz} = 0, \quad \mathbf{n} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}, \quad \mathbf{p}_n = \begin{bmatrix} p_{nx} \\ p_{ny} \\ p_{nz} \end{bmatrix}$$

Using the parametric line equation, one may write

$$\mathbf{p}_n = \mathbf{p} + \mathbf{n}t = \begin{bmatrix} p_x + N_1 t \\ p_y + N_2 t \\ p_z + N_3 t \end{bmatrix}$$

Solving for t

$$N_0 + N_1(p_x + N_1 t) + N_2(p_y + N_2 t) + N_3(p_z + N_3 t) = 0, \quad t = -\frac{N_0 + N_1 p_x + N_2 p_y + N_3 p_z}{N_1^2 + N_2^2 + N_3^2}$$

Note that this procedure takes care of topic 15. in section **2.1 Descriptive Geometry Topics**, Projection (normal) of a point onto a plane.

3 Principal Lines and Planes

In all the following cases the lines are assumed to be on two points P and Q given by their position vectors $\mathbf{p}$ and $\mathbf{q}$.

A line on two points with the same z-coordinate is *horizontal*. It is therefore called a principal line because it is on a horizontal plane $z\{-Z_0 : 0 : 0 : 1\}$, a *principal* plane normal to the z-axis.

$$\mathbf{z} = \mathbf{p} + (\mathbf{q} - \mathbf{p})t, \quad \mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ Z_0 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_x \\ q_y \\ Z_0 \end{bmatrix}$$

A line on two points with the same y-coordinate is *frontal*. It is therefore called a principal line because it is on a frontal plane $z\{-Y_0 : 0 : 1 : 0\}$, a *principal* plane normal to the y-axis.

$$\mathbf{y} = \mathbf{p} + (\mathbf{q} - \mathbf{p})t, \quad \mathbf{p} = \begin{bmatrix} p_x \\ Y_0 \\ p_z \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_x \\ Y_0 \\ q_z \end{bmatrix}$$

A line on two points with the same x-coordinate is *profile*. It is therefore called a principal line because it is on a profile plane $x\{-X_0 : 1 : 0 : 0\}$, a *principal* plane normal to the x-axis.

$$\mathbf{z} = \mathbf{p} + (\mathbf{q} - \mathbf{p})t, \quad \mathbf{p} = \begin{bmatrix} X_0 \\ p_y \\ p_z \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} X_0 \\ p_y \\ q_z \end{bmatrix}$$

4 Plane Definitions

Essentially, coefficients (homogeneous coordinates) of implicit equation of plane n are defined by three points, say, P, Q, R, *e.g.*, $P\{1 : p_x : p_y : p_z\}$.

$$n\{N_0 : N_1 : N_2 : N_3\} \equiv \left\{ \begin{vmatrix} p_x & p_y & p_z \\ q_x & q_y & q_z \\ r_x & r_y & r_z \end{vmatrix} : - \begin{vmatrix} 1 & p_y & p_z \\ 1 & q_y & q_z \\ 1 & r_y & r_z \end{vmatrix} : \begin{vmatrix} p_x & 1 & p_z \\ q_x & 1 & q_z \\ r_x & 1 & r_z \end{vmatrix} : - \begin{vmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ r_x & r_y & 1 \end{vmatrix} \right\}$$

4.1 Thinking Geometrically

You probably realize that the scalar, implicit equation of a plane is just

$$N_0x_0 + N_1x_1 + N_2x_2 + N_3x_3 = 0$$

where, conventionally, the homogeneous variable point coordinate x_0 is set $x_0 = 1$ and

$$\mathbf{n} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

is a free vector normal to the plane surface. But what about N_0? The shortest distance to the origin from the plane is given by

$$d_n = \frac{N_0}{\sqrt{N_1^2 + N_2^2 + N_3^2}}$$

If $d_n < 0$ then, in order to reach the origin from the plane, one must proceed in the sense *opposite* to that given by the direction of $\mathbf{n}$. With these insights we can regard the plane with the same intuitive knowledge concerning where it is and how is it tilted as we can visualize where a point is located, with respect to the chosen origin, from its three given coordinates.

4.2 Alternate Plane Definitions

There are other ways to specify a plane such as

- a given pair of intersecting lines, including parallel and principal trace pairs,
- a given line and point,
- a given *slope* line that always implies the plane on a family of parallel horizontal lines that are normal to and intersect the slope line or
- a pair of intersecting lines, called *principal traces*, each on either of two of the three principal *coordinate* planes, $z = 0, \ \ y = 0, \ \ x = X_0$.

In effect, all of these reduce to two intersecting lines that reduce to three convenient points. As an exercise in geometric thinking, try to visualize all of these situations.

5 Slope, Bearing and True Length of Line Segment $P \to Q$

- *Slope* is the angle ϕ that the line on PQ makes with a horizontal plane.

$$\tan\phi = \frac{q_z - p_z}{\sqrt{(q_x - p_x)^2 + (q_y - p_y)^2}}$$

- *Bearing* is the clockwise angle θ in the range $0 \ \ldots \ 2\pi$ that the horizontal projection of PQ makes with the horizontal direction called *north*. For convenience, on most maps, that direction is given by the vector

$$north = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

  This can be a little tricky to sort out so let us introduce angle θ' and tabulate rules to remove ambiguity.

$$\theta' = \tan^{-1}\left\|\frac{q_y - p_y}{q_x - p_x}\right\| = \tan^{-1}\left\|\frac{r_y}{r_x}\right\|$$

  Depending on the sign of r_y and r_x, θ is determined as follows.

r_x	r_y	
$+$	$+$	$\theta = \frac{\pi}{2} - \theta'$
$+$	$-$	$\theta = \frac{\pi}{2} + \theta'$
$-$	$+$	$\theta = \frac{3\pi}{2} + \theta'$
$-$	$-$	$\theta = \frac{3\pi}{2} - \theta'$

  This convention is called *north-azimuth* angle. If the north-arrow is not "up-the-page" or if bearing is to be specified in the quadrants as NθE, NθW, SθE or SθW (these are the four legitimate ones, you cannot say, *e.g.*, WθN) you will have to devise an algorithm to convert standard north-azimuth to offset north-azimuth or quadrant convention.

- *True length* auxiliary projection of PQ appears on any plane n on which the points are located by position vectors $\mathbf{p}_n$ and $\mathbf{q}_n$ and the plane normal vector is $\mathbf{n}$ when

$$\mathbf{n} \cdot (\mathbf{q} - \mathbf{p}) = 0$$

  The length of the line segment is obviously

$$\sqrt{(q_x - p_x)^2 + (q_y - p_y)^2 + (q_z - p_z)^2}$$

6 Point or End View of Line

A point or end view of a line appears on any auxiliary projection plane n anywhere when

$$\mathbf{n} \times (\mathbf{q} - \mathbf{p}) = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

7 Line or Edge View of Plane

A line or edge view of a plane $m\{M_0 : M_1 : M_2 : M_3\}$ appears on any auxiliary projection plane n anywhere when

$$\mathbf{m} \cdot \mathbf{n} = \mathbf{0}, \quad \mathbf{m} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

8 True View or Shape of Plane

True shape of a plane m is available on any auxiliary projection plane n where all lines, at least those on three points PQR, where $P \in m$, $Q \in m$, $R \in m$, appear in true length, *i.e.*,

$$\mathbf{m} \times \mathbf{n} = \mathbf{0}$$

Note that, by necessity, m and n are parallel or in our "typewriter shorthand" $m(=)n$.

9 Slope, Strike and Dip of Plane

These properties are important when surveying hidden, *e.g.*, underground, features like sedimentary plane strata containing valuable minerals.

- *Downward slope* of plane n is the angle ϕ_n.

  $$\phi_n = \cos^{-1}\left[\mathrm{sgn}(N_3)\frac{\mathbf{n}}{\|\mathbf{n}\|} \cdot \frac{\mathbf{z}}{\|\mathbf{z}\|}\right], \quad \mathrm{sgn}(N_3) = N_3/\sqrt{N_3^2}$$

  See section **3** for $\mathbf{z}$. Note definition of the "sign" function, sgn.

- *Strike* is the bearing of any horizontal line on n.

  $$\tan\theta' = \frac{h_y}{h_x}, \quad \mathbf{h} = \begin{bmatrix} h_x \\ h_y \\ 0 \end{bmatrix} = \mathbf{z} \times \mathrm{sgn}(N_3)\mathbf{n}$$

  Strike can be taken in either sense. The sign assigned to these direction numbers is not important.

- *Dip* is the bearing of any *slope line* or line of steepest slope on n.

  $$\tan\theta' = \frac{d_y}{d_x}, \quad \mathbf{d} = \begin{bmatrix} d_x \\ d_y \\ 0 \end{bmatrix} = \mathbf{h} \times \mathbf{z}$$

  Dip is taken in the descending sense, *i.e.*, $P \to Q$ where $q_z \le p_z$.

See section **5** to convert θ' to θ, a conventional bearing angle.

10 Principal Piercing Points of Line

Where is point X, with position vector $\mathbf{x}$, where the line on PQ intersects plane z, the horizontal principal coordinate plane? Parameterizing the line

$$\mathbf{x} = \mathbf{p} + (\mathbf{q} - \mathbf{p})t = \begin{bmatrix} p_x + (q_x - p_x)t \\ p_y + (q_y - p_y)t \\ p_z + (q_z - p_z)t \end{bmatrix} = \begin{bmatrix} x_x \\ x_y \\ 0 \end{bmatrix}, \quad t = \frac{p_z}{p_z - q_z}$$

Now repeat the exercise to find Y where PQ intersects the principle frontal plane. Study Fig. 4 on p.12, section **2.7** and Fig. 5 on p.15, section **2.7.4**.

11 Lines Parallel and Perpendicular to Planes and Other Lines

Study Fig. 3 on p.10 and sections **2.6.5** and **2.6.6** on p.11.

- Line PQ is parallel (=) to plane n if

$$\mathbf{n} \cdot (\mathbf{q} - \mathbf{p}) = 0$$

- Line PQ is normal (+) to plane n if

$$\mathbf{n} \times (\mathbf{q} - \mathbf{p}) = \mathbf{0}$$

- Line PQ is parallel (=) to line RS if

$$(\mathbf{q} - \mathbf{p}) \times (\mathbf{s} - \mathbf{r}) = \mathbf{0}$$

- Line PQ is normal (+) to line RS if

$$(\mathbf{q} - \mathbf{p}) \cdot (\mathbf{s} - \mathbf{r}) = 0$$

12 Planes Mutually Parallel and Perpendicular

Planes m and n are parallel (=) if $\mathbf{m} \times \mathbf{n} = \mathbf{0}$. They are normal if $\mathbf{m} \cdot \mathbf{n} = 0$.

13 Point on Plane

Point p is on plane n, $P \in n$, if

$$\begin{bmatrix} 1 \\ p_x \\ p_y \\ p_z \end{bmatrix} \cdot \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = 0$$

14 Normal Distance from Point to Plane

See section **2**, above. Study projection in *auxiliary view*. Given point P and plane n, the shortest, normal distance is $\|\mathbf{p}-\mathbf{p}_n\|$.

15 Intersection of Line and Plane

Study Fig. 8 on p.19, section **2.8.5**. Find point S on given line PQ and plane n.

$$S = (PQ) \cap n, \quad S \in PQ, \quad S \in n$$

If the plane coordinates are given use the parametric line equation and the plane equation.

$$\mathbf{s} = \mathbf{p} + (\mathbf{q}-\mathbf{p})t = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix}, \quad \begin{bmatrix} 1 \\ s_x \\ s_y \\ s_z \end{bmatrix} \cdot \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = 0$$

If the the plane is on three given points ABC then solve three parametric equations for t, u, v and get $S\{1 : s_x : s_y : s_z\}$ from the first or last one.

$$\begin{aligned} \mathbf{s} &= \mathbf{p} + (\mathbf{q}-\mathbf{p})t \\ \mathbf{d} &= \mathbf{b} + (\mathbf{c}-\mathbf{b})u \\ \mathbf{s} &= \mathbf{a} + (\mathbf{d}-\mathbf{a})v \end{aligned}$$

16 Normal Projection of Point on Plane

See section **2** and auxiliary view projection.

17 Angle between Line and Plane

Given line PQ and plane n, ϕ, the angle between them is given by

$$\phi = \sin^{-1}\left[\frac{\mathbf{n}}{\|\mathbf{n}\|} \cdot \frac{\mathbf{q}-\mathbf{p}}{\|\mathbf{q}-\mathbf{p}\|}\right]$$

18 Angle between Lines

Given two lines PQ and RS, ϕ, the angle between them is given by

$$\phi = \cos^{-1}\left[\frac{\mathbf{q}-\mathbf{p}}{\|\mathbf{q}-\mathbf{p}\|} \cdot \frac{\mathbf{s}-\mathbf{r}}{\|\mathbf{s}-\mathbf{r}\|}\right]$$

19 Line of Intersection between Planes

Assume the two planes m and n are given by their equation coefficients (homogeneous coordinates). First a plane $o\{O_0 : O_1 : O_2 : O_3\}$ normal to both m and n, conveniently chosen on the origin so $O_0 = 0$, is determined. Then the other coefficients, that comprise the normal vector $\mathbf{o}$, can be immediately written.

$$\mathbf{o} = \begin{bmatrix} O_1 \\ O_2 \\ O_3 \end{bmatrix} = \mathbf{m} \times \mathbf{n}$$

The intersection of these three planes must be a point $P = m \cap n \cap o$ on the line of intersection (LoX). Another point Q can be found by choosing another plane o' parallel to o. This is done by augmenting the normal vector coordinates with some conveniently chosen $O'_0 \neq 0$ and finding $Q = m \cap n \cap o'$. To find the distance between o and o', hence the distance between P and Q, find the normal projection of the origin O on o'. Recall how that is done by referring to section **2** and auxiliary views, once again. If one is content with a single point parametrization of (LoX) then

$$\mathbf{s} = \mathbf{p} + (\mathbf{m} \times \mathbf{n})t$$

can be written with $\mathbf{s}$ being the position vector of any point on (LoX). To find P, *i.e.*, $\mathbf{p}$, proceed in the same way as the the plane n was determined on given points PQR in section **4**, above, except using *plane* coordinates.

$$P \left\{ \begin{vmatrix} O_1 & O_2 & O_3 \\ M_1 & M_2 & M_3 \\ N_1 & N_2 & N_3 \end{vmatrix} : - \begin{vmatrix} 0 & O_2 & O_3 \\ M_0 & M_2 & M_3 \\ N_0 & N_2 & N_3 \end{vmatrix} : \begin{vmatrix} 0 & O_1 & O_3 \\ M_0 & M_1 & M_3 \\ N_0 & N_1 & N_3 \end{vmatrix} : - \begin{vmatrix} 0 & O_1 & O_2 \\ M_0 & M_1 & M_2 \\ N_0 & N_1 & N_2 \end{vmatrix} \right\}$$

$$P\{p_0 : p_1 : p_2 : p_3\} \quad \rightarrow \quad \mathbf{p} = \begin{bmatrix} p_1/p_0 \\ p_2/p_0 \\ p_3/p_0 \end{bmatrix}$$

20 Dihedral Angle between Planes

The angle ϕ between planes m and n is given as

$$\phi = \cos^{-1} \left[\frac{\mathbf{m}}{\|\mathbf{m}\|} \cdot \frac{\mathbf{n}}{\|\mathbf{n}\|} \right]$$

21 Joining Skew Lines

This topic is treated in detail on pp.66-71, section **7**.

22 Intersection and Development of surfaces

This some elements of this topic is treated on pp.45-55, section **5**. There is a digression, on pp.51-52, section **5.6**, into an important application of sections **17** and **18**, above.

23 Vector Statics

Particular attention is drawn to the Maxwell diagram analysis of planar, pin-jointed structures that uses an elegant adaptation of the “method of joints” treated in most elementary texts on mechanics while omitting the Maxwell diagram approach. This is found on pp.28-30, section **3.3**. Note that the “method of sections” that the student is urged to use because it is a way to evaluate specific members in a panel-point loaded, pin-jointed planar truss without having to find the force in all members. Maxwell’s diagram makes complete evaluation fast and fun and a closure error check is included for free.

24 Earthworks

Alas, nothing appears on this interesting and useful topic in **Fundamentals of Geometric Construction**. It would be sad if at least one important idea were not mentioned here. Imagine a proposed excavation for a building foundation on an undeveloped site on irregular terrain. The horizontal projection of the pit is mapped, say, as a rectangle divided into a grid of squares. From the topological contours that appear as curves of constant elevation, “spot” levels on all grid square corners are obtained and every square is augmented by *one* diagonal. Since the desired elevation of the excavation “floor” has been chosen, vertical heights between floor and the grid corners can be calculated. Taking two sides of each square and the diagonal joining the open corners of the two there is a prism of triangular cross-section extending from the ground surface to the floor. The entire volume of the excavation is calculated by summing the volumes v_i of all such prisms. Assume the triangle on the surface has vertices ABC and the corresponding ones on the floor are $A'B'C'$. To obtain the volume the average height h is multiplied by the normal section area a_i of the prism. This may be the area of $A'B'C'$ but not if the floor has various slopes and discontinuities. The points ABC (or $A'B'C'$) must be projected on a horizontal plane as $A''B''C''$ to obtain a. Draw a neat, labelled free-hand sketch to illustrate and clarify this concept.

$$v_i = \left(\frac{h_A + h_B + h_C}{3}\right) a_i, \quad h_A = a_z - a'_z, \quad h_B = b_z - b'_z, \quad h_C = c_z - c'_z, \quad a_i = \frac{1}{2}\begin{vmatrix} 1 & a_x & a_y \\ 1 & b_x & b_y \\ 1 & c_x & c_y \end{vmatrix}$$

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