MECH 576
Geometry in Mechanics

November 30, 2009

Kinematics of Clavel's "Delta" Robot

1 The DELTA Robot

"DELTA", a three dimensional translational manipulator, appears below in Fig. 1.

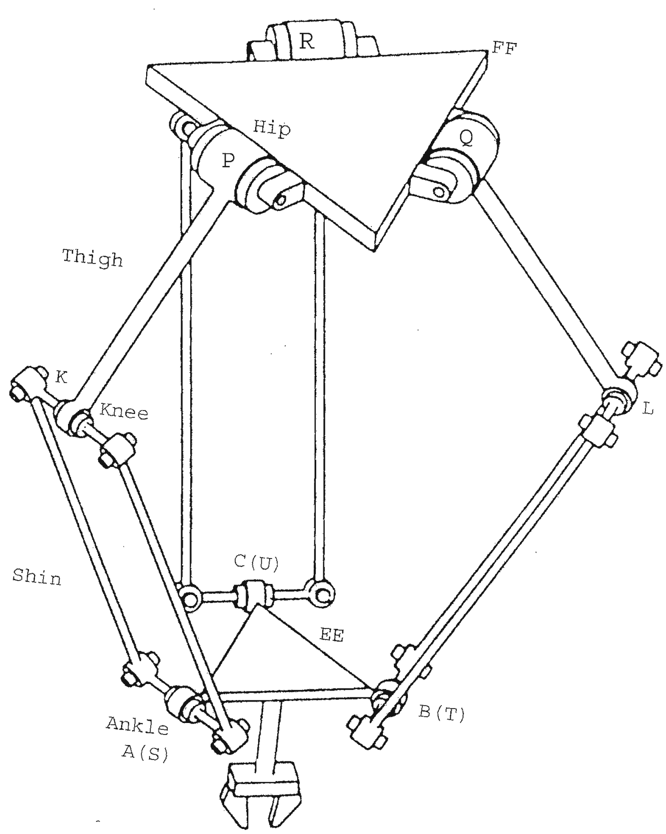

Figure 1: Symmetrical (Conventional) DELTA Robot

In what follows, inverse and direct kinematic analysis procedures –IK and DK– will be outlined. These are both based essentially on an intersection-of-line-and-sphere conceptual model and a simple translational mapping. The manipulator consists of a base or fixed frame FF connected to a moving frame or end effector EE via three legs. Starting at FF there are the motorized (actuated) "hips" P, Q, R that move the rigid "thighs" through angles θ, φ, ψ, respectively. It may be convenient to measure these as clockwise, thighs moving downwards in a right hand screw sense as one proceeds around FF in the sequence P, Q, R and taking zero as thighs horizontal like a ballerina doing the "splits". These three angles are to be determined in the inverse problem. This places the "knees" at K, L, M. These positions are given in the case of the direct problem. The "shins", that connect knees to "ankles" A, B, C when referring to the mapping "home" position and S, T, U when referring to the mapped final displacement position, are Π-joints (parallelogram) whereon all points move on spheres. In IK the centres are S, T, U and K, L, M move on the surface. In DK the centres are K, L, M and S, T, U move on the surface.

2 Procedure: Inverse Kinematics (IK)

A method will be outlined to establish knee positions K, L, M given the ankle positions S, T, U. As opposed to direct kinematics, each leg is treated separately. Knowing the ankle position S, define sphere α.

$$\alpha: \ (x_1 - s_1)^2 + (x_2 - s_2)^2 + (x_3 - s_3)^2 - s = 0 \tag{1}$$

Knowing the hip position P define sphere β, centred on P and radius $KP = \sqrt{p}$, and plane γ on K and P and containing the absolute point $Z\{0:0:0:1\}$ so that γ is normal to the hip axis at P. Together, β and γ define a circle that intersects α.

$$\begin{aligned} \beta: \ (x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2 - p = 0 \\ \gamma: \ G_0 + G_1 x_1 + G_2 x_3 + G_3 x_3 = 0 \end{aligned} \tag{2}$$

Subtracting sphere equations β from α gives a second plane.

$$\alpha - \beta: \ 2(p_1 - s_1)x_1 + 2(p_2 - s_2)x_2 + 2(p_3 - s_3)x_3 - (p_1^2 + p_2^2 + p_3^2) + (s_1^2 + s_2^2 + s_3^2) + p - s = 0 \tag{3}$$

Proceeding with α, Eq. 1, γ, the second of Eq. 2 and Eq. 3 substituted for Eqs. 13, 14 and 15, below, one solves for x_i, the two possible positions for K. This procedure must be repeated for the other two legs and one must select judiciously between the solutions. *I.e.*, usually one should choose the more "bow-legged" rather than "knock-kneed" stance for the legs. This also applies, more or less, to direct kinematic solutions.

3 Procedure: Direct Kinematics (DK)

A method will be outlined to establish the positions of the three anchor points or ankles, initially placed ideally on EE at points $A(0,0,0)$, $B(b_1,0,0)$, $C(c_1,c_2,0)$. These are moved onto the three spheres, σ, τ, υ, in FF, that are swept when the three actuators at the hips, P, Q, R, put the sphere centres on points $K(k_1,k_2,k_2)$, $L(l_1,l_2,l_3)$, $M(m_1,m_2,m_3)$ on the "knees" and the anchor points on the other end of the shins trace the spheres of known, respective radii, $\sqrt{s}, \sqrt{t}, \sqrt{u}$. The necessary translation is accomplished by a simple point *translation* matrix $[\mathbf{T}]$. The transformation puts $A \rightarrow S, B \rightarrow T$, $C \rightarrow U$, where $S(s_1,s_2,s_3), T(t_1,t_2,t_3), U(u_1,u_2,u_3)$ are points on the respective spheres.

- Three sphere equations are

$$\sigma: \ (s_1 - k_1)^2 + (s_2 - k_2)^2 + (s_3 - k_3)^2 - s = 0 \tag{4}$$
$$\tau: \ (t_1 - l_1)^2 + (t_2 - l_2)^2 + (t_3 - l_3)^2 - t = 0 \tag{5}$$
$$\upsilon: \ (u_1 - m_1)^2 + (u_2 - m_2)^2 + (u_3 - m_3)^2 - u = 0 \tag{6}$$

- Subtracting Eqs. 5 and 6 from the Eq. 4 produces the following two linear equations, Eqs. 7 and 8, that may then be solved simultaneously with, say, Eq. 4.

$$\begin{aligned} \sigma - \tau: \ & 2(l_1 t_1 - k_1 s_1) + 2(l_2 t_2 - k_2 s_2) + 2(l_3 t_3 - k_3 s_3) \\ & + k_1^2 - l_1^2 + k_2^2 - l_2^2 + k_3^2 - l_3^2 + t - s = 0 \end{aligned} \tag{7}$$
$$\begin{aligned} \sigma - \upsilon: \ & 2(m_1 u_1 - k_1 s_1) + 2(m_2 u_2 - k_2 s_2) + 2(m_3 u_3 - k_3 s_3) \\ & + k_1^2 - m_1^2 + k_2^2 - m_2^2 + k_3^2 - m_3^2 + u - s = 0 \end{aligned} \tag{8}$$

- Move $A \rightarrow S$, $B \rightarrow T$, $C \rightarrow U$ via the translation vector $\mathbf{x}$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad [\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ x_3 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

$$[\mathbf{T}] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{s} \end{bmatrix} \tag{10}$$

$$[\mathbf{T}] \begin{bmatrix} 1 \\ b_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 + b_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{t} \end{bmatrix} \tag{11}$$

$$[\mathbf{T}] \begin{bmatrix} 1 \\ c_1 \\ c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 + c_1 \\ x_2 + c_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{u} \end{bmatrix} \tag{12}$$

- Substitute the expressions for vectors $\mathbf{s}$, $\mathbf{t}$, $\mathbf{u}$, *i.e.*, the last three elements of the vectors in Eqs. 10, 11 and 12, in Eqs. 4, 7 and 8 and solve simultaneously for x_1, x_2, x_3 then compute the actual positions of S, T, U with Eqs. 10, 11 and 12. Eqs. 4, 7 and 8 become, after substitution, Eqs. 13, 14 and 15.

$$\sigma: \ (x_1 - k_1)^2 + (x_2 - k_2)^2 + (x_3 - k_3)^2 - s = 0 \tag{13}$$

$$\begin{aligned} \sigma - \tau: \ & 2(l_1 - k_1 - b_1)x_1 + 2(l_2 - k_2)x_2 + 2(l_3 - k_3)x_3 \\ & + k_1^2 + k_2^2 + k_3^2 - l_2^2 - l_3^2 - (b_1 - l_1)^2 - s + t = 0 \end{aligned} \tag{14}$$

$$\begin{aligned} \sigma - \upsilon: \ & 2(m_1 - k_1 - c_1)x_1 + 2(m_2 - k_2 - c_2)x_2 + 2(m_3 - k_3)x_3 \\ & + k_1^2 + k_2^2 + k_3^2 - (c_1 - m_1)^2 - (c_2 - m_2)^3 - m_3^2 - s + u = 0 \end{aligned} \tag{15}$$

- The advantage of this method is that it does not assume a symmetric manipulator. Upper and lower platform triangles and hip and shin lengths can be chosen arbitrarily. However the design must be configured so that all "hip", "knee" and "ankle" R-joint axes are horizontal, *i.e.*, normal to vertical direction x_3.

- Simultaneous solution of Eqs. 13, 14 and 15 can be advantageously done by *parametrization* of the line of intersection of planes $\sigma - \tau$, Eq. 14 and $\sigma - \upsilon$, Eq. 15. This avoids elimination of coordinate variables, say, x_1 and x_2, and produces a quadratic univariate in parameter v directly. This is explained in detail after Fig. 1.

Given the sphere and platform data tabulated in Fig. 2 two translation vectors $\mathbf{x}$ and $\mathbf{x}'$ are found by substituting these data into Eqs. 13, 14 and 15 and solving for x_1, x_2, x_3. The numerical versions of these equations appear as Eqs. 16.

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 - 9 = 0, \ \ x_1 - x_2 - x_3 - \frac{5}{3} = 0 \\ 6x_1 + 4(3 - \sqrt{3})x_2 + 2x_3 - [4(3 - \sqrt{3})^2 + 15] = 0 \end{aligned} \tag{16}$$

4 Intersection of 3 Spheres with a Parameterized Line

Compress coefficients of the two plane equations, Eq. 14 and Eq. 15, according to Eq. 17.

$$E_0 + E_1 x_1 + E_2 x_2 + E_3 x_3 = 0, \ \ F_0 + F_1 x_1 + F_2 x_2 + F_3 x_3 = 0 \tag{17}$$

Then expand the six properly signed 2×2 minor determinants of the 2×4 matrix of the plane coordinates, *i.e.*, coefficients, E_i, F_i to establish the axial Plücker coordinates G_{ij} of their intersecting line $\mathcal{G}_a$.

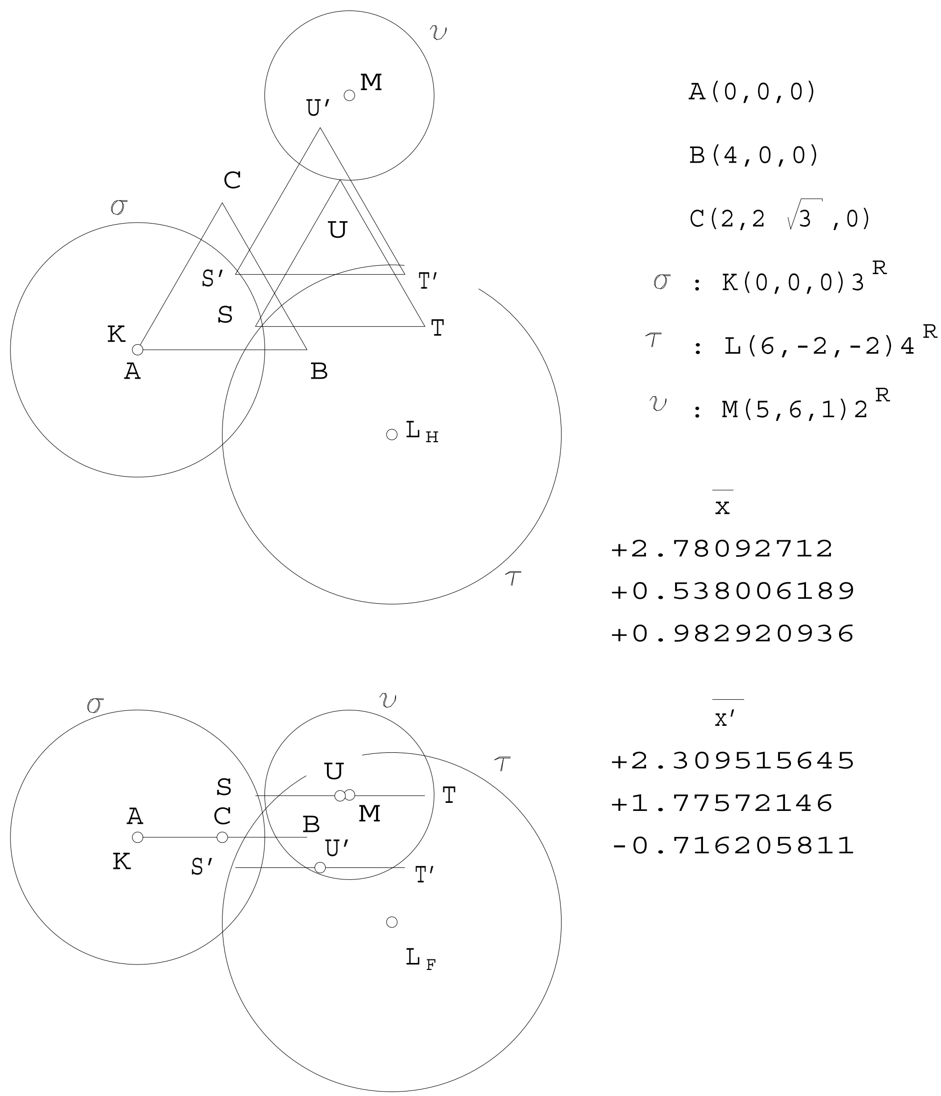

Figure 2: "Home" and Two Solution Poses of EE on a Robot with Asymmetric Thighs and Shins

$$\mathcal{G}_a : \begin{vmatrix} E_0 & E_1 & E_2 & E_3 \\ F_0 & F_1 & F_2 & F_3 \end{vmatrix} \rightarrow \{G_{01} : G_{02} : G_{03} : G_{23} : G_{31} : G_{12}\}$$
$$\rightarrow \{E_0F_1 - E_1F_0 : E_0F_2 - E_2F0 : E_0F_3 - E_3F_0 : E_2F_3 - E_3F_2 : E_3F_1 - E_1F_3 : E_1F_2 - E_2F_1\} \quad (18)$$

These transformations and others to follow are explained in some detail in [3]. Parametrization begins on a base point on $P\{p_0 : p_1 : p_2 : p_3\}$ on line $\mathcal{G}$. P may be conveniently chosen on the intersection of $\mathcal{G}$ with its normal plane $p\{P_0 : P_1 : P_2 : P_3\}$ on the origin. Since *piercing point* of line with plane relations are usually formulated with *radial* line coordinates $\mathcal{G}(g_{ij}$ these too are expressed below. in Eq. 19.

$$p\{P_0 : P_1 : P_2 : P_3\} \equiv \{0 : G_{23} : G_{31} : G_{12}\} \equiv \{0 : g_{01} : g_{02} : g_{03}\}$$
$$\begin{array}{llllllllll} p_0 = & & +g_{01}P_1 & +g_{02}P_2 & +g_{03}P_3 & = & G_{23}P_1 & +G_{31}P_2 & +G_{12}P_3 \\ p_1 = & -g_{01}P_0 & & +g_{12}P_2 & -g_{31}P_3 & = & & G_{03}P_2 & -G_{02}P_3 \\ p_2 = & -g_{02}P_0 & -g_{12}P_1 & & +g_{23}P_3 & = & -G_{03}P_1 & & +G_{01}P_3 \\ p_3 = & -g_{03}P_0 & +g_{31}P_1 & -g_{23}P_2 & & = & G_{02}P_1 & -G_{01}P_2 & \end{array} \quad (19)$$

Now with dehomogenized coordinates of P and the normalized direction numbers of $\mathcal{G}_a$ the line can be parameterized in v as shown in Eq. 20.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} p_1/p_0 \\ p_2/p_0 \\ p_3/p_0 \end{bmatrix} + \begin{bmatrix} G_{23}v/\sqrt{G_{23}^2 + G_{31}^2 + G_{12}^2} \\ G_{31}v/\sqrt{G_{23}^2 + G_{31}^2 + G_{12}^2} \\ G_{12}v/\sqrt{G_{23}^2 + G_{31}^2 + G_{12}^2} \end{bmatrix} \tag{20}$$

Substituting for x_i in Eq. 13 from Eq. 20 leaves a quadratic in v. Returning the two values of v to Eq. 20 yields the two desired sets of piercing point coordinates. Placing the end effector (moving platform) EE artificially with A on the origin, B along the x_1-axis and C in the plane $x_3 = 0$ and adding the vector $\mathbf{x}$ to each, causes all to arrive on their respective spheres at S, T, U. Those who wish to read more, about the mathematical background of some of the previous calculations, are referred to Klien's little book [1] and may be inclined to browse the introductory lecture on the site mentioned in [3].

5 Parametrization of Linear Implicit Equations

Many first courses in calculus, like [2], introduce parametric equations by way of reducing given ones to implicit form and by taking derivatives of parametric functions. It is not easy to find treatment of the inverse problem, *i.e.*, how to parameterize. Although an example has been already given one may delve a little deeper by considering the simpler two dimensional parametrization of a line. Consider Fig. 3 that shows the line $\mathcal{G}$ represented by the line g. As in the previous case the parametric origin is point $P = g \cap P$ where p is a line through O and normal to g. It is easy to see that these lines and the parametric vector equation of g can be written as shown in Eq. 21.

$$g: \ -12x_0 + 3x_1 + 4x_2 = 0, \ \ p: \ 0x_0 + 4x_1 - 3x_2 = 0, \ \ \mathbf{x} = \mathbf{p} + \mathbf{q}v \tag{21}$$

The vector equation expands to Eq. 22.

$$\mathbf{x} = \begin{bmatrix} x_1/x_0 \\ x_2/x_0 \end{bmatrix} = \begin{bmatrix} 36/25 \\ 48/25 \end{bmatrix} + \begin{bmatrix} 4v/5 \\ -3v/5 \end{bmatrix} \tag{22}$$

Geometrically the parametrization process is a *projection*. This can be seen by comparing the lower and upper graphs in Fig. 3. Below, the two Cartesian coordinates $x_1/x_0, x_2/x_0$ are related by an implicit equation. It does not matter which variable is chosen as "independent". The equation may be solved explicitly in both ways. Above, $x_1/x_0 = x$ and $x_2/x_0 = y$ is plotted separately against a true independent variable v that runs from P along the *real number line*. Notice the dehomogenization of point coordinates

$$\left| \begin{array}{rrr} x_0 & x_1 & x_2 \\ 0 & 4 & -3 \\ -12 & 3 & 4 \end{array} \right| \rightarrow \{25 : 36 : 48\} \rightarrow P\left(\frac{36}{25}, \frac{48}{25}\right)$$

and normalization of the direction vector $\mathbf{q}$.

$$\mathbf{q} = \begin{bmatrix} 4/\sqrt{4^2 + (-3)^2} \\ -3/\sqrt{4^2 + (-3)^2} \end{bmatrix}$$

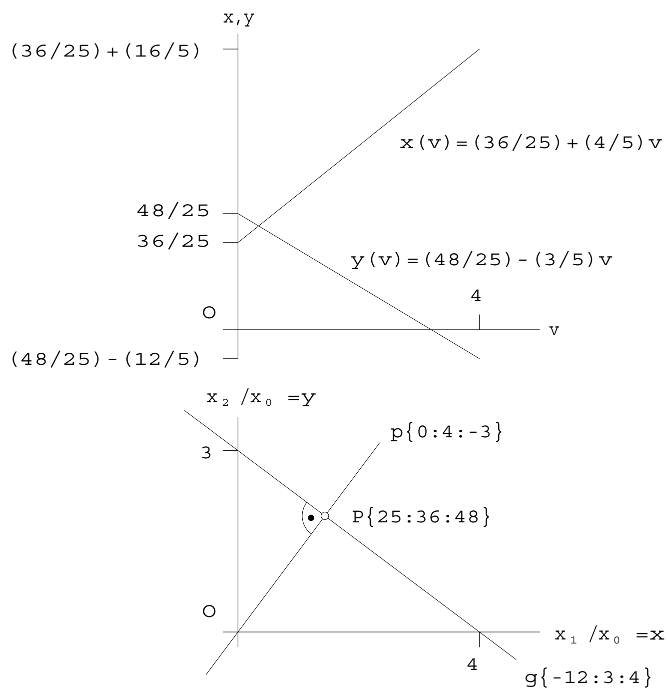

Figure 3: Parameterizing a Planar Line

6 The Planar 2-Legged Π-Jointed Translational Manipulator

In what follows the inverse and direct kinematics of a simple two-legged fully parallel, *i.e.*, only one activated joint per leg, translational robot will be exposed. This device may be thought of as a planar version of "Delta". One may refer to either Fig. 4 or Fig. 5 to follow the subsequent introductory description.

- The mechanism itself consists of two draughting machinelike legs each composed of a pair of parallelogram 4-bar linkages shown to be connected by six revolute or R-joints.
- Each leg spans between FF, the fixed frame and EE the end effector. FF_1 and FF_2 just refers to one leg or the other. They represent the same base.
- The actuated R-joints are taken to be base-mounted, the ones closest to the labels FF_1 and FF_2. Their angular excursions appear in Fig. 5 as arc-arrows and are the red dots in Fig. 4.
- The system is assumed to be in an arbitrary "home" position indicated for both legs by the line on the two R-Joints on each labeled EE_H. Similarly an arbitrary "operating point" on EE is labeled OP_H in this home position.

6.1 Inverse Kinematics

In inverse kinematics the given "target" operating point is labeled OP_T and the required translation vector joins OP_H to OP_T. This vector is applied to all four R-joints shown on EE_H to move these to both instances of EE_T. With these in place the intersections of the blue circles, centred on the blue dots on both legs, with the red circles centred on the red actuated R-joints established the two solution configurations and hence the actuated joint angles can be determined straightaway. The computation is simple and not worthy of explanation.

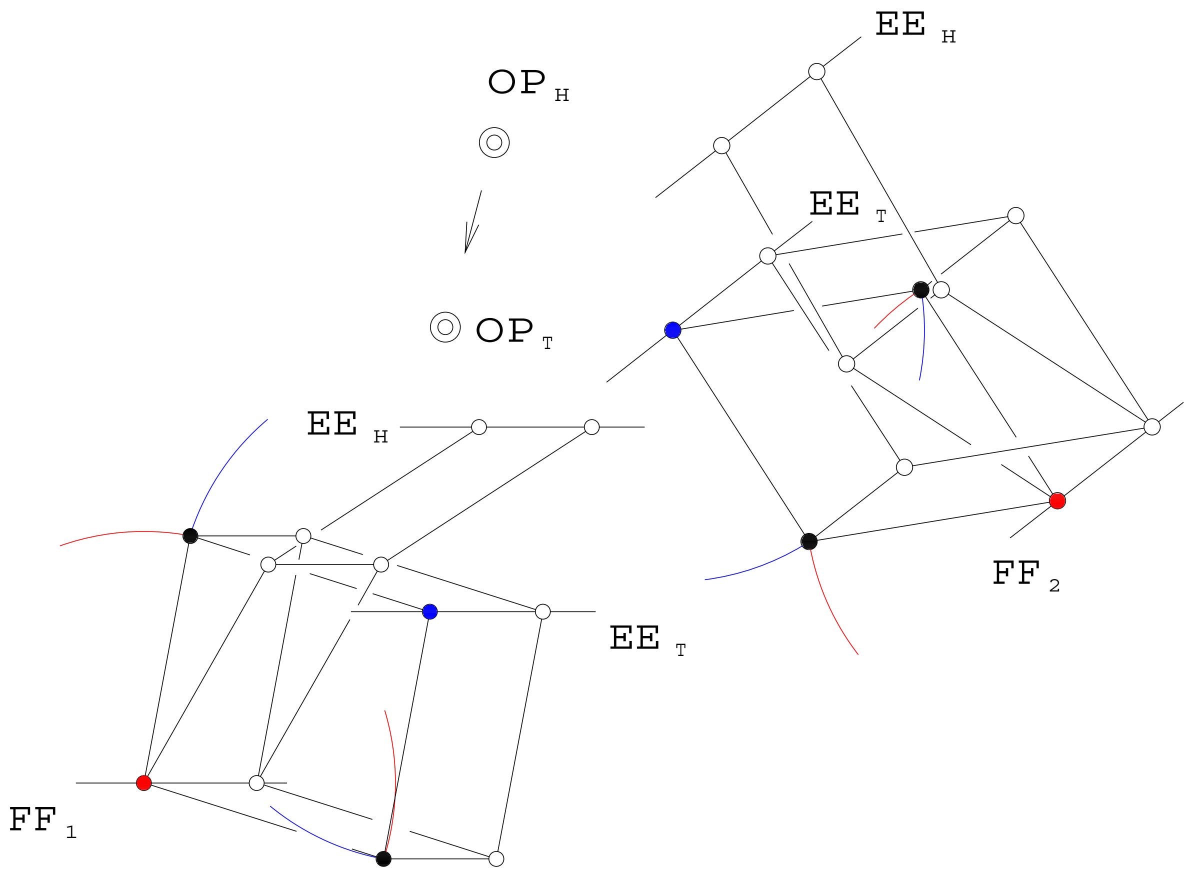

Figure 4: Inverse Kinematics

6.2 Direct Kinematics (Constructive)

For direct kinematics exactly the same setup is used except in this case OP_T are the unknowns to be determined and the joint angles at FF_1 and FF_2 are given. That means that the red dots now represent the red circle centres that are "fixed" by virtue of the given R-joint angles. The trick is to put the coupler vector $\overrightarrow{VO}$ so V lies on the right hand circle and O on the left while preserving the orientation of VO. This is accomplished by translating the right hand red centre dot, via a vector $\overrightarrow{VO}$, to the the blue one, along with the red circle which also becomes the blue one. Again the intersection of two circles gives two solutions representing alternate end effector poses EE_T and $\mathrm{EE}_{T'}$. After completing the rest of the left hand parallelogram, mapping backward with the vector $\overrightarrow{OV}$ completes the missing right hand parallelogram. Of course OP_T moves with EE_T and $\mathrm{OP}_{T'}$ with $\mathrm{EE}_{T'}$.

6.3 Direct Kinematics (Analytic)

For those averse to descriptions of geometric constructions in the plane, recall Eqs. 4 through 15 as these apply to "Delta". Here the three spheres are replaced by two circles, the red ones in Fig. 5. Their equations are

$$(x_1 - m_1)^2 + (x_2 - m_2)^2 - r^2 = 0, \;\; (y_1 - m_1)^2 + (y_2 - m_2)^2 - s^2 = 0 \tag{23}$$

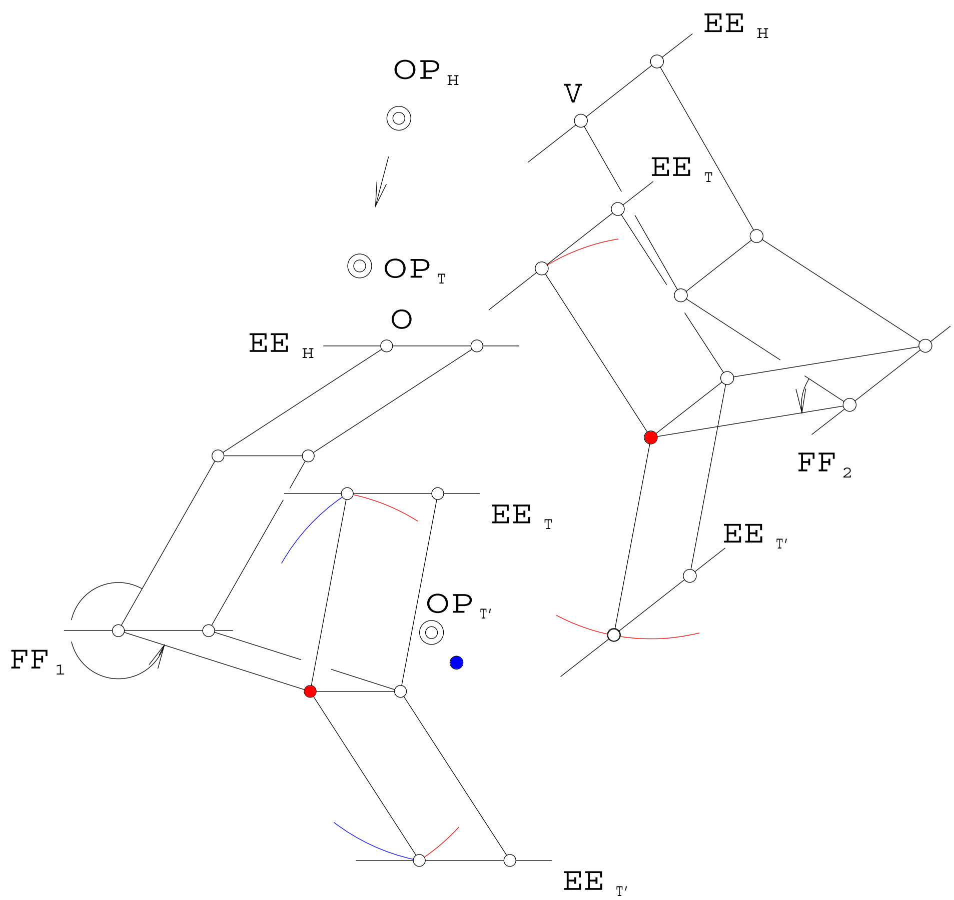

Figure 5: Direct Kinematics

The three points A, B, C on the end effector ankles are replaced by $O(0,0), V(0,v)$ and the mapping is

$$\begin{bmatrix} 1 \\ u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} 1 \\ u_1+v \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ u_1 & 1 & 0 \\ u_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ v \\ 0 \end{bmatrix} \tag{24}$$

and the two vectors on the left replace (x_1, x_2) and (y_1, y_2) and the two circles, Eq. 23, become Eq. 25.

$$(u_1 - m_1)^2 + (u_2 - m_2)^2 - r^2 = 0, \;\; (u_1 + v - n_1)^2 + (u_2 - n_2)^2 - s^2 = 0 \tag{25}$$

The transformation leaves the first circle essentially unchanged but the second has been translated by $\overrightarrow{VO}$. Taking the difference between Eqs. 25 produces the linear equation Eq. 26, the line of intersection between the two circles in u_1, u_2.

$$2(n_1 - m_1 - v)u_1 + 2(n_2 - m_2)u_2 + (m_1^2 + m_2^2 - n_1^2 - n_2^2 + 2n_1 v + s^2 - r^2) = 0 \tag{26}$$

Solving the first of Eqs. 25 with Eq. 26 yields a quadratic in, say, u_1 and each value with Eq. 26 gives the two required points of intersection between the circles shown in Fig. 5 which essentially solves the problem.

6.4 The Wrong Way

Assume one proceeds, instead, using a 4-bar mechanism model by assuming that the red circles are connected by a coupler whose length and orientation are given by OV. The upper right circle is taken as origin centred and points on it are parameterized on the angle θ that indicates crank rotation. Tangent of the coupler orientation angle is t. Three constraint equations can be written as follows. The first is the coupler length constraint where $Y(y_1, y_2)$ is a point on the lower left circle of radius s, r is radius of the other and $L = |OV|$.

$$\begin{aligned} (y_1 - r\cos\theta)^2 + (y_2 - r\sin\theta)^2 - L^2 = 0 \\ (y_1 - n_1)^2 + (y_2 - n_2)^2 - s^2 = 0 \\ (y_1 - r\cos\theta)t - y_2 + r\sin\theta = 0 \end{aligned} \tag{27}$$

Taking advantage of all algebraic simplification possibilities and making the customary tangent half angle substitutions for $\cos\theta$ and $\sin\theta$ results in a quartic univatiate in $\tan\frac{\theta}{2}$. Coding this as a symbolic computation and then substituting the values

$$L = 12.337, \;\; n_1 = -17.619, \;\; n_2 = 0, \;\; r = s = 8.341, \;\; t = \tan 5.51^\circ$$

from Fig. 5 returns four values for the crank angle as

$$\tan\frac{\theta}{2} = -1.791, \;\; -0.04983 \pm 1.8743i, \;\; 1.1229$$

The last, real value corresponds to the upper pose in Fig. 5. The first does not correspond to the other one. Evidently, there are undiscovered errors and readers are encouraged to look for them.

References

[1] Klein, F.C. (2004) *Elementary Mathematics from an Advanced Standpoint -Geometry-*, Dover, ISBN 0-486-43481-8.

[2] Peterson, T.S. (1950) *Elements of Calculus*, Harper & Brothers, New York, pp.114–118.

[3] Zsombor-Murray, P.J. (1999) "Grassmannian Reduction of Quadratic Forms", *Lecture* **I**, *MECH 576 – Geometry in Mechanics*, <`http://www.cim.mcgill.ca/∼paul/GRQF69f.pdf`>