MECH 576
Geometry in Mechanics

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Solving Simultaneous Multivariate Polynomial Equations

1 Explicit Solutions of Cubic and Quartic Univariates

Although explicit solutions to third and fourth order polynomial equations in one variable exist, *e,g.*, see [1], they are rarely discussed in detail, let alone presented for implimentation as coded algorithms. Even commercial computational packages like “Maple” seem to produce solutions iteratively through the application of numerical methods in a conceptually concise “top-down” way. For example, upon examining the polynomial solution procedure used by “Matlab”, one finds a few lines of very high-level code, calling forth computation of companion matrix eigenvalues. No doubt, delving into that procedure and beyond would lead to a formidable “tree climbing” exercise. These methods produce succinct multilevel description and robust code which accurately finds all roots. However it is not clear that this is most suitable for real time control of manipulators whose kinematics may require no equations of order beyond the fourth. Furthermore one finds real roots sometimes accompanied by a small imaginary residue. Examination of the classical methods of Ferrari and Cardano was stimulated by work done in connection with [3], a recent analysis of a one dof three-legged platform mechanism, supported on one C-S and two R-S legs, actuated by a single R-joint. Determination of the third S-joint position completes the solution and this is done with a coplanar intersection of circle and ellipse. It will be seen that the implementation of explicit cubic and quartic equation solving is detered not by the amount of calculation required but by decisions which need to be taken to select the appropriate method, in the case of a cubic, or four out of eight solutions, in the case of a quartic.

2 The Cubic

The cubic equation in the single variable x,

$$x^3 + ax^2 + bx + c = 0 \tag{1}$$

where a, b and c are given, real coefficients, has three roots $x_i,\ i = 1, 2, 3$. All three may be real or there may be a pair of complex conjugate ones.

2.1 Reduction

All equations like Eq. 1 may be reduced to three terms, *i.e.*, the term ax^2 is removed by substituting $x = y - \frac{a}{3}$ in Eq. 1 and collecting constant coefficients.

$$y^3 + py + q = 0 \tag{2}$$

Note that

$$p = -\frac{a^3}{3} + b, \quad q = 2\left(\frac{a}{3}\right)^3 - \frac{ab}{3} + c$$

2.2 Cardano's Solution

If

$$Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \leq 0$$

then Eq. 2 has three roots $y_{1,2,3}$; one real root and a complex conjugate pair.

$$y_1 = A + B, \quad y_{2,3} = -\frac{A+B}{2} \pm i\frac{A-B}{2}\sqrt{3} \tag{3}$$

A and B are calculated as follows.

$$A = \left(\sqrt{Q} - \frac{q}{2}\right)^{\frac{1}{3}}, \quad B = \left(-\sqrt{Q} - \frac{q}{2}\right)^{\frac{1}{3}} \tag{4}$$

2.3 Trigonometric Solution

If

$$Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 < 0$$

then Eq. 2 has three real roots $y_{1,2,3}$.

$$y_1 = 2\sqrt{\frac{-p}{3}}\cos\frac{\alpha}{3}, \quad y_{2,3} = -2\sqrt{-\frac{p}{3}}\cos\left(\frac{\alpha \pm \pi}{3}\right) \tag{5}$$

where

$$\cos\alpha = -\frac{q}{2\sqrt{-\left(\frac{p}{3}\right)^3}}$$

2.4 Implementation

The listing CUSA 01-03-26 shows a simple procedure to compute the three roots of a cubic. If $Q < 0$ then 42 FLOPS are needed to find the three real roots. This increases to 43 if $\cos\alpha < 0$ making it necessary to replace α with its supplement $\pi - \alpha$. The only other "side-step" operation that was deemed necessary occurs if $\cos\alpha = 0$ and

$$\alpha = \tan^{-1}\left(\frac{\sin\alpha}{\cos\alpha}\right)$$

returns a "division by zero" error. This was avoided by setting $\alpha = (\pi/2)$ and testing $\cos\alpha$ after it is computed but before the division in $\tan^{-1}$ is performed. Maybe the sign $(-)$ of q in the numerator of the $\cos\alpha$ computation should be used to change $\alpha = (\pi/2) \to (-\pi/2)$ in this case. Verification of this possibility has not been carried out. It is pointed out so that the user may implement this or some other more appropriate measure if problems arise here. If $Q \geq 0$ the three roots can be determined in only 31 FLOPS.

```
 100
INPUT A,B,C:PI=3.141592654:REM PIC\CUSA 01-03-26 110
A3=A/3:P=B-A*A3:Q=2*A3^3*B+C:P3=P/3:AL=PI/2:X=1/3:
    QQ=P3^3+Q*Q/4:IF QQ>=0 THEN GOTO 160
120 P3=-P3:CA=-Q/(2*SQR(P3^3):IF CA=0 THEN GOTO 140 130
SA=SQR(1-CA^2):AL=ATN(SA/CA):IF CA<0 THEN CA=PI-AL 140
CA1=COS(AL/3):CA2=COS((AL+PI)/3):CA3=((AL-PI)/3) 150
T=2*SQR(P3):X1=T*CA1-A3:X2=-T*CA2-A3:X3=-T*CA3-A3:
    PRINT X1,X2,X3:STOP
160 SQ=SQR(QQ):Q2=-Q/2:AA=(Q2+SQ)^X:BB=(Q2-SQ)^X:
    AB=AA+BB:BA=AA-BB
170 X1=AB-A3:X2=-AB/2-A3:X3=BA*SQR(3)/2
    PRINT X1,X2;"+/-I";X3:STOP:END
```

3 The Quartic

The quartic equation in the single variable z,

$$z^4 + b_3 z^3 + b_2 z^2 + b_1 z + b_0 = 0 \tag{6}$$

where $b_i,\ i = 0, 1, 2, 3$ are given, real coefficients, has four roots z_i. All may be real or there may be one or two pairs of complex conjugate ones.

3.1 The Cubic Resolvent

A cubic *resolvent* equation, Eq. 7, can be expressed in terms of coefficients of Eq. 6.

$$u^3 - b_3 u^2 + (b_1 b_3 - 4b_0)u - (b_1^2 + b_0 b_3^2 - 4b_0 b_2) = 0 \tag{7}$$

Its real root u_1, the greatest of three if all are real, *i.e.*, $u_1 \geq u_2 \geq u_3$, is chosen and used to produce all eight roots of the following quadratic by evaluating the root pair for each of four choices indicated by the two ($\pm$) signs. The variable in Eq. 8 is called v to remind us that four of these "candidate" solutions are spurious; they do not fulfil Eq. 6.

$$v^2 + \left[\frac{b_3}{2} \pm \sqrt{\left(\frac{b_3}{2}\right)^2 + u_1 - b_2}\right] v + \left[\frac{u_1}{2} \pm \sqrt{\left(\frac{u_1}{2}\right)^2 - b_0}\right] = 0 \tag{8}$$

It appears, though no proof was encountered nor were exhaustive tests conducted, that if one solution in a pair of candidates is valid, then so is the other. Therefore if this conjecture is valid, only ${}_4C_2 = 6$ tests are needed to select the four valid z_i and not ${}_8C_4 = 70$. This should not incur too much computational overhead. Note that a representative of all four root pairs must be examined for minimal residue, regardless of what sort of test is chosen. There is no guarantee of nullity here.

3.2 The Test

An obvious but crude policy is to substitute four candidate roots into Eq. 6, compare the four residues and select the root pairs associated with the least two. Another way is to see how closely Eq. 9, the factored version of Eq. 6, can reproduce the four coefficients b_i. Eqs. 10 were implemented, assuming that these would constitute a satisfactory test. This group of second-term product residues is simpler to compute than the other three possible partial tests which are all sums-of-products.

$$(z - v_i)(z - v_{i+1})(z - v_j)(z - v_{j+1}) = 0 \tag{9}$$

$$\begin{gathered} v_1v_2v_3v_4 - b_0 = r_1, \;\; v_1v_2v_5v_6 - b_0 = r_2, \;\; v_1v_2v_7v_8 - b_0 = r_3, \\ v_3v_4v_5v_6 - b_0 = r_4, \;\; v_3v_4v_7v_8 - b_0 = r_5, \;\; v_5v_6v_7v_8 - b_0 = r_6 \end{gathered} \tag{10}$$

But, alas, this was not to be. After frustrating results with false selections and even "ties" with small residue among more than two result pairs, the "brute-force two best result pairs" criterion was finally used at a cost of about 136 FLOPS which is nearly as many as are required to produce all eight candidate roots.

3.3 Coding the Quartic

Once again one begs the reader's indulgence to consider the following routine coded in `GW-BASIC`. It computes and selects the four roots, working its way through the necessary decision network, ensuring hard zero imaginary residue for definite (as opposed to "close to") real roots.

```
 100 DIM V(8,2),S(4,2):PI=3.141592654:AL=PI/2
110 INPUT B3,B2,B1,B0:B4=4*B0:A=-B2:B=B1*B3-B4 120
C=B2*B4-B1^2-B0*B3^2:A3=A/3:P=B-A*A3:Q=2*A3^3-A3*B+C 130
P3=P/3:QQ=P3^3+Q*Q/4:IF QQ>=0 THEN GOTO 200 140
P3=-P3:CA=-Q/(2*SQR(P3^3)):IF CA=0 THEN GOTO 160 150
SA=SQR(1-CA^2):AL=ATN(SA/CA):IF CA<0 THEN AL=PI-AL 160
CA1=COS(AL/3):CA2=COS((AL+PI)/3):CA3=COS(((AL-PI)/3) 170
T=2*SQR(P3):U1=T*CA1-A3*:X2=-T*CA2-A3:X3=-T*CA3-A3 180 IF X2>U1 THEN
U1=X2:IF X3>U1 THEN U1=X3 190 GOTO 220 200
SQ=SQR(QQ):Q2=-Q/2:AA=Q2+SQ:BB=Q2-SQ:X=1/3 210
GA=SGN(AA):GB=SGN(BB):U1=GA*(ABS(AA))^X+GB*(ABS(BB))^X-A3 220
BT=B3/2:R1=BT^2+U1-B2:IF R1<0 THEN STOP 230
R1=SQR(R1):UT=U1/2:R2=UT^2-B0:IF R2<0 THEN STOP 240
R2=SQR(R2):M=1:LPRINT:LPRINT B3,B2,B1,B0 250 FOR J=1 TO 2 260 FOR
K=1 TO 2 270 F=((-1)^J)*R1-BT:G=((-1)^K)*R2-UT:H=F*F+4*G:VR=F/2 280
VI=SQR(ABS(H))/2:IF H<0 THEN GOTO 300 290
V1R=VR+VI:V2R=VR-VI:V1I=0:V2I=0:GOTO 310 300
V1R=VR:V2R=VR:V1I=VI:V2I=-VI 310
V(M,1)=V1R:V(M,2)=V1I:M=M+1:V(M,1)=V2R:V(M,2)=V2I:M=M+1 320 NEXT K
330 NEXT J 340 FOR I=1 TO 7 STEP 2:REM *** Select 4 valid roots ***
350 VR1=V(I,1):VI1=V(I,2):VR2=VR1^2-VI1^2:
    VI2=2*VR1*VI1:VR3=VR1*VR2-VI1*VI2
```

```
360 VI3=VR1*VI2+VR2*VI1:VR4=VR2^2-VI2^2:
    VI4=2*VR2*VI2:J=(I+1)/2:S(J,2)=I
370 S(J,1)=(VR4+B3*VR3+B2*VR2+B1*VR1+B0)^2
    +(VI4+B3*VI3+B2*VI2+B1*VI1+B0)^2
380 NEXT I 390 FOR I=1 TO 3 400 K=4-I 410 FOR J=1 TO K 420 IF
S(J,1)<=S(J+1,1) THEN GOTO 440 430
T=S(J,1):S(J,1)=S(J+1,1):S(J+1,1)=T
    T=S(J,2):S(J,2)=S(J+1,2):S(J+1,2)=T
440 NEXT J 450 NEXT I 460 I=S(1,2):J=S(2,2):LPRINT
"Real","Imaginary" 470 LPRINT V(I,1),V(I,2):LPRINT V(I+1,1),V(I+1,2)
    LPRINT V(J,1),V(J,2):LPRINT V(J+1,1),V(J+1,2)
480 STOP:END
```

4 Solving Simultaneous Polynomial Equations

Often, a viable alternative to formulating linear differential equations to solve a problem is to formulate it as a sufficient set of simultaneous higher order equations. To solve these one must reduce the set to a single univariate polynomial, of some still higher order, and then evaluate its roots.

4.1 Constraints

Consider two simultaneous second order homogeneous equations in the plane. The coefficients are those which appear in the quadratic form

$$[w\ x\ y]\begin{bmatrix} a_{11} & a_{12} & a_{31} \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}\begin{bmatrix} w \\ x \\ y \end{bmatrix} = 0$$

Two such equations appear below.

$$a_{11}w^2 + 2a_{12}wx + 2a_{31}yw + a_{22}x^2 + 2a_{23}xy + a_{33}y^2 = 0 \tag{11}$$

$$b_{11}w^2 + 2b_{12}wx + 2b_{31}yw + b_{22}x^2 + 2b_{23}xy + b_{33}y^2 = 0 \tag{12}$$

These may be combined to yield a fourth order univariate in, say, x. Then four roots can be obtained explicitly according to the method and algorithm outlined in section **3**. Unique corresponding values of y are computed with one of two auxiliary parabolic equations which are formed as differences between Eqs. 11 and 12, premultiplied by a complementary pair of zeroth and first order Bezout multipliers.

4.2 Bezout's Method

Salmon [2] describes an efficient method to reduce a pair of homogeneous polynomial equations, of order n in m variables, to a single equation of order n^2 in $m-1$ variables, *i.e.*, wherein one of the variables has been eliminated. A system of n auxiliary equations of order $n-1$ is generated from the original, given pair of polynomials and the $n \times n$ determinant of coefficients of this system must vanish. All variables except the one chosen to be eliminated remain embedded in the coefficients. Examine the pair of higher order equations below.

$$a_0 y^n w^0 + a_1 y^{n-1} w^1 + a_2 y^{n-2} w^2 + \cdots + a_{n-2} y^2 w^{n-2} + a_{n-1} y^1 w^{n-1} + a_n y^0 w^n = 0 \tag{13}$$

$$b_0 y^n w^0 + b_1 y^{n-1} w^1 + b_2 y^{n-2} w^2 + \cdots + b_{n-2} y^2 w^{n-2} + b_{n-1} y^1 w^{n-1} + b_n y^0 w^n = 0 \tag{14}$$

Note that the general structure of the conics expressed in Eqs. 11 and 12 has been preserved to make the procedure easier to follow. Only y and w appear. The latter is a homogenizing coordinate which is set $w = 1$ unless one seeks results on the line $w = 0$ which closes the projective plane. The variable x and any others, which do not exist in the case of conics, are absorbed in a_i and b_i. The pair Eqs. 13 and 14 are transformed to the system of $n - 1^{th}$ order equations by multiplying them with $n-1$ pairs of complementary Bezout multipliers and then forming the difference of the multiplied pair. The variable product $y^n w^0$ does not appear in the difference. Here are the first four pairs of multipliers and the last one.

$$m_0 \begin{cases} b_0 y^0 w^0 \\ a_0 y^0 w^0 \end{cases}$$

$$m_1 \begin{cases} b_0 y^1 w^0 + b_1 y^0 w^1 \\ a_0 y^1 w^0 + a_1 y^0 w^1 \end{cases}$$

$$m_2 \begin{cases} b_0 y^2 w^0 + b_1 y^1 w^1 + b_2 y^0 w^2 \\ a_0 y^2 w^0 + a_1 y^1 w^1 + a_2 y^0 w^2 \end{cases}$$

$$m_3 \begin{cases} b_0 y^3 w^0 + b_1 y^2 w^1 + b_2 y^1 w^2 + b_3 y^0 w^3 \\ a_0 y^3 w^0 + a_1 y^2 w^1 + a_2 y^1 w^2 + a_3 y^0 w^3 \end{cases}$$

$$m_{n-1} \begin{cases} b_0 y^{n-1} w^0 + b_1 y^{n-2} w^1 \ \cdots + b_{n-2} y^1 w^{n-2} + b_{n-1} y^0 w^{n-1} \\ a_0 y^{n-1} w^0 + a_1 y^{n-2} w^1 \ \cdots + a_{n-2} y^1 w^{n-2} + a_{n-1} y^0 w^{n-1} \end{cases}$$

4.3 The Fourth Order Univariate of a Conic Pair

Choosing to eliminate y, Eqs. 11 and 12 are rewritten to absorb x in A_i and B_i and rehomogenized.

$$A_0 y^2 + A_1 y w + A_2 w^2 = 0, \quad B_0 y^2 + B_1 y w + B_2 w^2 = 0$$

Multiplying these by B_0 and A_0, forming the difference to yield a first equation, then multiplying by $B_0 y + B_1 w$ and $A_0 y + A_1 w$ and forming the difference to yield a second equation produces the following pair linear in y.

$$(A_0 B_1 - B_0 A_1) y + (A_0 B_2 - B_0 A_2) w = 0, \quad (A_0 B_2 - B_0 A_2) y + (A_1 B_2 - B_1 A_2) w = 0$$

Here, $n = 2$ and the eliminating determinant or *eliminant* is

$$\begin{vmatrix} A_0 B_1 - B_0 A_1 & A_0 B_2 - B_0 A_2 \\ A_0 B_2 - B_0 A_2 & A_1 B_2 - B_1 A_2 \end{vmatrix} = 0$$

Note that

$$A_0 = a_{33},\ \ B_0 = b_{33},\ A_1 = 2(a_{31}w + a_{23}x),\ \ B_1 = 2(b_{31}w + b_{23}x)$$

$$A_2 = a_{11}w^2 + 2a_{12}wx + a_{22}x^2,\ \ B_2 = b_{11}w^2 + 2b_{12}wx + b_{22}x^2$$

Coefficients for the quartic in x can be easily coded and computed with only 70 FLOPS.

$$Ax^4 + Bx^3 + Cx^2 + Ex + F = 0$$

where

$$A = k_2k_3 - k_4^2,\ \ c_1 = k_6 + k_7,\ \ B = k_1k_3 + k_2c_1 - 2k_4k_5,\ \ c_2 = k_9 + k_{10}$$

$$C = k_1c_1 + k_2c_2 - 2k_4k_8 - k_5^2,\ \ E = k_1c_2 + k_2k_{11} - 2k_5k_8,\ \ F = k_1k_{11} - k_8^2$$

and

$$k_1 = 2(b_{33}a_{31} - a_{33}b_{31})w,\ \ k_2 = 2(b_{33}a_{23} - a_{33}b_{23}),\ \ k_3 = 2(b_{23}a_{22} - a_{23}b_{22})$$

$$k_4 = b_{33}a_{22} - a_{33}b_{22},\ \ k_5 = 2(b_{33}a_{12} - a_{33}b_{12})w,\ \ k_6 = 2(b_{31}a_{22} - a_{31}b_{22})w$$

$$k_7 = 4(b_{23}a_{12} - a_{23}b_{12})w,\ \ k_8 = (b_{33}a_{11} - a_{33}b_{11})w^2,\ \ k_9 = 4(b_{31}a_{12} - a_{31}b_{12})w^2$$

$$k_{10} = 2(b_{23}a_{11} - a_{23}b_{11})w^2,\ \ k_{11} = 2(b_{31}a_{11} - a_{31}b_{11})w^3$$

4.4 Cubic and Conic Intersection

The following example is slightly more complicated. A general planar cubic curve is specified as

$$a_{11}w^3 + a_{12}w^2x + a_{13}yw^2 + a_{21}wx^2 + a_{22}y^2w + a_{23}x^3 + a_{31}x^2y + a_{32}xy^2 + a_{33}y^3 = 0 \qquad (15)$$

Then Eq. 15 is paired with Eq. 12 and the surviving variable x is gathered up and embedded and one sets $w = 1$, as before. As regards Eq. 16, note the rearrangement of coefficients.

$$a_{33}y^3 + (a_{22} + a_{32}x)y^2 + (a_{13} + a_{31}x^2)y + (a_{11} + a_{12}x + a_{21}x^2 + a_{23}x^3) = 0 \qquad (16)$$

The two equations to be solved are simplified as

$$a_0y^3 + a_1y^2 + a_2y + a_3 = 0,\ \ b_0y^3 + b_1y^2 + b_2y = 0$$

Note that the conic has been converted into a pseudo-cubic without a "constant" term by multiplying Eq. 12 by y. Applying Bezout multipliers m_0 and m_1 to the pair above, forming differences and augmenting with a third equation Eq. 12, already of second order, produces the required 3×3 eliminant of coefficients of three cubics reduced to conics.

$$\begin{vmatrix} b_0 & b_1 & b_2 \\ a_0b_1 - b_0a_1 & a_0b_2 - b_0a_2 & a_3b_0 \\ a_0b_2 - b_0a_2 & a_1b_2 - a_2b_1 - a_3b_0 & a_3b_1 \end{vmatrix} = 0$$

Expanding this determinant and collecting on powers of x yields the sixth order univariate whose coefficients are a bit too unwieldy to express here explicitly.

$$Ax^6 + Bx^5 + Cx^4 + Ex^3 + Fx^2 + Gx + H = 0$$

This must be solved numerically for six values of y. How can these be paired with unique vaules of x? Assume one had chosen to eliminate x, rather than y. Then the collection of coefficients would have produced the following pair of equations.

$$a_{23}x^3 + (a_{21} + a_{31}y)x^2 + (a_{12} + a_{32}y^2)x + (a_{11} + a_{13}y + a_{22}y^2 + a_{33}y^3) = 0$$

$$b_{22}x^2 + 2(b_{12} + b_{23}y)x + (b_{11} + 2b_{31}y + b_{33}y^2) = 0$$

Simplifying

$$a_0^*x^3 + a_1^*x^2 + a_2^*x + a_3^* = 0, \quad b_0^*x^2 + b_1^*x + b_2^* = 0$$

Taking these two equations, multiplying the second by x to form a pseudo-cubic, applying Bezout multiplier m_0 and forming the difference produces a quadratic in x^2. Remember, at this time y is known. A second quadratic is immediately available as the second conic. Applying Bezout multiplier m_0 again, this pair yields an equation linear in x.

4.5 General Backsubstitution

One is not always so lucky. There is no immediately obvious path to linear backsubstitution when dealing with a general system of three simultaneous quadrics with two remaining unknowns. For this reason it is worthwhile to examine a simple elimination procedure which leads to desired results. Here is the system in X_1 and X_2 with some value of X_3 absorbed in the coefficients.

$$a_{ij}\{X_1^2 \;\; X_2^2 \;\; X_1X_2 \;\; X_1 \;\; X_2 \;\; 1\}^T = \{0 \;\; 0 \;\; 0\}^T$$

or

$$a_{11}X_1^2 + a_{12}X_2^2 + a_{13}X_1X_2 + a_{14}X_1 + a_{15}X_2 + a_{16} = 0 \tag{17}$$
$$a_{21}X_1^2 + a_{22}X_2^2 + a_{23}X_1X_2 + a_{24}X_1 + a_{25}X_2 + a_{26} = 0 \tag{18}$$
$$a_{31}X_1^2 + a_{32}X_2^2 + a_{33}X_1X_2 + a_{34}X_1 + a_{35}X_2 + a_{36} = 0 \tag{19}$$

First $a_{21},\ a_{31},\ a_{32}$ are eliminated from Eqs. 18 and 19 to produce Eqs. 20 and 21.

$$b_{22}X_2^2 + b_{23}X_1X_2 + b_{24}X_1 + b_{25}X_2 + b_{26} = 0 \tag{20}$$
$$c_{33}X_1X_2 + c_{34}X_1 + c_{35}X_2 + c_{36} = 0 \tag{21}$$

$$b_{2j} = a_{21}a_{1j} - a_{11}a_{2j}, \quad j = 2,\ \ldots,\ 6$$

$$c_{3j} = b_{32}b_{2j} - b_{22}b_{3j}, \quad j = 3,\ \ldots,\ 6$$

$$b_{3j} = a_{31}a_{1j} - a_{11}a_{3j}, \quad j = 2,\ \ldots,\ 6$$

Substituting for X_1 in Eqs. 17 and 20 with results from Eq. 21 yields a quartic, Eq. 22, and a cubic, Eq. 23, univariate in X_2.

$$d_4X_2^4 + d_3X_2^3 + d_2X_2^2 + d_1X_2 + d_0 = 0 \tag{22}$$
$$e_3X_2^3 + e_2X_2^2 + e_1X_2 + e_0 = 0 \tag{23}$$

$$d_4 = a_{12}c_{33}^2, \quad d_3 = (2a_{12}c_{34} - a_{13}c_{35} + a_{15}c_{33})c_{33}, \quad a_c = a_{14}c_{35} + a_{13}c_{36}$$

$$d_2 = (a_{16}c_{33} - a_c + 2a_{15}c_{34})c_{33} + (a_{12}c_{34} - a_{13}c_{35})c_{34} + a_{11}c_{35}^2$$
$$d_1 = (a_{15}c_{34} - a_c + 2a_{16}c_{33})c_{34} + (2a_{11}c_{35} - a_{14}c_{33})c_{36}$$
$$d_0 = (a_{11}c_{36} - a_{14}c_{34})c_{36} + a_{16}c_{34}^2$$
$$e_3 = b_{22}c_{33}, \quad e_2 = b_{22}c_{34} - b_{23}c_{35} + b_{25}c_{33}$$
$$e_1 = b_{25}c_{34} + b_{26}c_{33} - b_{23}c_{36} - b_{24}c_{35}, \quad e_0 = b_{26}c_{34} - b_{24}c_{36}$$

Raising Eq. 23 to a quartic, premultiplying by initial coefficients and subtracting Eq. 22 gives another cubic.

$$f_3X_2^3 + f_2X_2^2 + f_1X_2 + f_0 = 0 \tag{24}$$
$$f_3 = e_3d_3 - d_4e_2, \quad f_2 = e_3d_2 - d_4e_1, \quad f_1 = e_3d_1 - d_4e_0, \quad f_0 = e_3d_0$$

The premultiplied difference between Eqs. 23 and 24 has eliminated X_2^3 in Eq. 25, below.

$$g_2X_2^2 + g_1X_2 + g_0 = 0 \tag{25}$$
$$g_2 = f_3e_2 - e_3f_2, \quad g_1 = f_3e_1 - e_3f_1, \quad g_0 = f_3e_0 - e_3f_0$$

Then X_2^3 is eliminated by raising Eq. 25 to a cubic and forming the appropriate difference with Eq. 24.

$$h_2X_2^2 + h_1X_2 + h_0 = 0 \tag{26}$$
$$h_2 = g_2f_2 - f_3g_1, \quad h_1 = g_2f_1 - f_3g_0, \quad h_0 = g_2f_0$$

Finally the required equation, Eq. 27, is obtained with the difference between Eqs. 25 and 26.

$$(h_2g_1 - g_2h_1)X_2 + (h_2g_0 - g_2h_0) = 0 \tag{27}$$

The last remaining step is to get X_1 with Eq. 21. This guarantees that evaluating a unique pair X_1 and X_2 will cost no more than 141 FLOPS.

4.6 Spherical Backsubstitution

Spherical direct kinematic backsubstitution begins with the set of three constraint equations, F_1 :, F_2 :, F_3 :, which are considerably simpler than Eqs. 17, 20 and 21, *i.e.*,

$$\begin{array}{llllll} F1: & a_1Y_1^2 & +a_2Y_2^2 & & & +a_{y1} = 0 \\ F2: & & d_2Y_2^2 & d_4Y_1Y_2 & & +d_{y1} = 0 \\ F3: & & & f_4Y_1Y_2 & +f_{y1}Y_1 \quad +f_{y2}Y_2 & +f_{y3} = 0 \end{array}$$

wherein Y_3 is absorbed in $a_{y1}, \ d_{y1}, \ f_{y1}, \ f_{y2}, \ f_{y3}$. Using the coefficient nomenclature of Eqs. 17, 20 and 21, one sees

$$a_{11} = a_1, \quad a_{12} = a_2, \quad a_{13} = a_{14} = a_{15} = 0, \quad a_{16} = a_{y1},$$
$$b_{22} = d_2, \quad b_{23} = d_4, \quad b_{24} = b_{25} = 0, \quad b_{26} = d_{y1}$$
$$c_{33} = f_4, \quad c_{34} = f_{y1}, \quad c_{35} = f_{y2}, \quad c_{36} = f_{y3}$$

so, in the nomenclature of Eqs. 22 and 23,

$$d_4 = a_{11}c_{33}^2, \;\; d_3 = 2a_{12}c_{33}c_{34}, \;\; a_c = 0, \;\; d_2 = a_{16}c_{33}^2 + a_{12}c_{34}^2 + a_{11}c_{35}^2$$

$$d_1 = 2(a_{16}c_{33}c_{34} + a_{11}c_{35}c_{36}), \;\; d_0 = a_{11}c_{36}^2 + a_{16}c_{34}^2$$

$$e_3 = b_{22}c_{33}, \;\; e_2 = b_{22}c_{34} - b_{23}c_{35}, \;\; e_1 = b_{26}c_{33}, \;\; e_0 = b_{26}c_{34}$$

Although these are more compact than the complete relations for $d_4 \; \ldots \; d_0$ and $e_3, \; \ldots \; e_0$ below Eqs. 22 and 23, these nine coefficients all survive and the elimination process to arrive at Eq. 27 is nevertheless identical to that outlined above for the case of general backsubstitution.

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