

MECH 576 Geometry in Mechanics

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Centre of a Hyperboloid of One Sheet via Line Geometry

1 Introduction

A unique hyperboloid of one sheet may be defined by ruling three given skew lines $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ with a one parameter set of lines that intersects all three. M , the centre point of this quadric, may be found on the intersection of three planes p, q, r such that

$$M = p \cap q \cap r, \quad p = \mathcal{P} \cap \mathcal{P}_{qr}, \quad q = \mathcal{Q} \cap \mathcal{Q}_{rp}, \quad r = \mathcal{R} \cap \mathcal{R}_{pq}$$

where $\mathcal{P}_{qr}, \mathcal{Q}_{rp}, \mathcal{R}_{pq}$ are the respective “parallel partners” of $\mathcal{P}, \mathcal{Q}, \mathcal{R}$. \mathcal{P}_{qr} intersects \mathcal{Q} and \mathcal{R} . \mathcal{Q}_{rp} intersects \mathcal{R} and \mathcal{P} . \mathcal{R}_{pq} intersects \mathcal{P} and \mathcal{Q} .

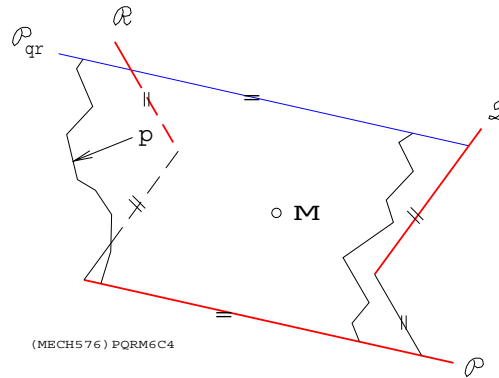


Figure 1: Relation of Parallel Partner \mathcal{P}_{qr} to Lines $\mathcal{P}, \mathcal{Q}, \mathcal{R}$

The approach outlined below makes extensive use of radial Plücker line coordinates. Axial coordinates would serve just as well.

2 Parallel Partners

Direction numbers of parallel partner lines are immediately available. One needs only to determine their moment vector.

$$\begin{aligned} \mathcal{P}_{qr} &\{p_{01} : p_{02} : p_{03} : p_{23}^{qr} : p_{31}^{qr} : p_{12}^{qr}\} \\ \mathcal{Q}_{rp} &\{q_{01} : q_{02} : q_{03} : q_{23}^{rp} : q_{31}^{rp} : q_{12}^{rp}\} \\ \mathcal{R}_{pq} &\{r_{01} : r_{02} : r_{03} : r_{23}^{pq} : r_{31}^{pq} : r_{12}^{pq}\} \end{aligned}$$

2.1 Homogeneous Constraints and Solutions

Parallel partners intersect on an absolute point. The partner in the opposite regulus intersects the other two given skew lines on Euclidean points. Multipliers λ , μ and ν appear in Eqs. 1,2,3,4 because homogeneous, not direction vector normalized, Plücker coordinates are being used.

$$\begin{aligned}
(p_{01}q_{23} + p_{02}q_{31} + p_{03}q_{12})\lambda + q_{01}p_{23}^{qr} + q_{02}p_{31}^{qr} + q_{03}p_{12}^{qr} &= 0 \\
(p_{01}r_{23} + p_{02}r_{31} + p_{03}r_{12})\lambda + r_{01}p_{23}^{qr} + r_{02}p_{31}^{qr} + r_{03}p_{12}^{qr} &= 0 \\
0\lambda + p_{01}p_{23}^{qr} + p_{02}p_{31}^{qr} + p_{03}p_{12}^{qr} &= 0
\end{aligned} \tag{1}$$

The first two of Eqs. 1 represent the intersection of \mathcal{P}_{qr} with \mathcal{Q} and \mathcal{R} , respectively. The third represents the Plücker condition imposed on \mathcal{P}_{qr} . The solution of three simultaneous equations in four homogeneous variables appears below as four determinants. λ is not required but is included to complete the solution.

$$\begin{aligned}
\lambda &= \begin{vmatrix} q_{01} & q_{02} & q_{03} \\ r_{01} & r_{02} & r_{03} \\ p_{01} & p_{02} & p_{03} \end{vmatrix}, \quad p_{23}^{qr} = - \begin{vmatrix} p_{01}q_{23} + p_{02}q_{31} + p_{03}q_{12} & q_{02} & q_{03} \\ p_{01}r_{23} + p_{02}r_{31} + p_{03}r_{12} & r_{02} & r_{03} \\ 0 & p_{02} & p_{03} \end{vmatrix} \\
p_{31}^{qr} &= \begin{vmatrix} p_{01}q_{23} + p_{02}q_{31} + p_{03}q_{12} & q_{01} & q_{03} \\ p_{01}r_{23} + p_{02}r_{31} + p_{03}r_{12} & r_{01} & r_{03} \\ 0 & p_{01} & p_{03} \end{vmatrix}, \quad p_{12}^{qr} = - \begin{vmatrix} p_{01}q_{23} + p_{02}q_{31} + p_{03}q_{12} & q_{01} & q_{02} \\ p_{01}r_{23} + p_{02}r_{31} + p_{03}r_{12} & r_{01} & r_{02} \\ 0 & p_{01} & p_{02} \end{vmatrix}
\end{aligned} \tag{2}$$

Similarly one may write the multipliers and moment vector elements for \mathcal{Q}_{rp} and \mathcal{R}_{pq} . Notice that $\lambda = \mu = \nu$.

$$\begin{aligned}
\mu &= \begin{vmatrix} r_{01} & r_{02} & r_{03} \\ p_{01} & p_{02} & p_{03} \\ q_{01} & q_{02} & q_{03} \end{vmatrix}, \quad q_{23}^{rp} = - \begin{vmatrix} q_{01}r_{23} + q_{02}r_{31} + q_{03}r_{12} & r_{02} & r_{03} \\ q_{01}p_{23} + q_{02}p_{31} + q_{03}p_{12} & p_{02} & p_{03} \\ 0 & q_{02} & q_{03} \end{vmatrix} \\
q_{31}^{rp} &= \begin{vmatrix} q_{01}r_{23} + q_{02}r_{31} + q_{03}r_{12} & r_{01} & r_{03} \\ q_{01}p_{23} + q_{02}p_{31} + q_{03}p_{12} & p_{01} & p_{03} \\ 0 & q_{01} & q_{03} \end{vmatrix}, \quad q_{12}^{rp} = - \begin{vmatrix} q_{01}r_{23} + q_{02}r_{31} + q_{03}r_{12} & r_{01} & r_{02} \\ q_{01}p_{23} + q_{02}p_{31} + q_{03}p_{12} & p_{01} & p_{02} \\ 0 & q_{01} & q_{02} \end{vmatrix}
\end{aligned} \tag{3}$$

$$\begin{aligned}
\nu &= \begin{vmatrix} p_{01} & p_{02} & p_{03} \\ q_{01} & q_{02} & q_{03} \\ r_{01} & r_{02} & r_{03} \end{vmatrix}, \quad r_{23}^{pq} = - \begin{vmatrix} r_{01}p_{23} + r_{02}p_{31} + r_{03}p_{12} & p_{02} & p_{03} \\ r_{01}q_{23} + r_{02}q_{31} + r_{03}q_{12} & q_{02} & q_{03} \\ 0 & r_{02} & r_{03} \end{vmatrix} \\
r_{31}^{pq} &= \begin{vmatrix} r_{01}p_{23} + r_{02}p_{31} + r_{03}p_{12} & p_{01} & p_{03} \\ r_{01}q_{23} + r_{02}q_{31} + r_{03}q_{12} & q_{01} & q_{03} \\ 0 & r_{01} & r_{03} \end{vmatrix}, \quad r_{12}^{pq} = - \begin{vmatrix} r_{01}p_{23} + r_{02}p_{31} + r_{03}p_{12} & p_{01} & p_{02} \\ r_{01}q_{23} + r_{02}q_{31} + r_{03}q_{12} & q_{01} & q_{02} \\ 0 & r_{01} & r_{02} \end{vmatrix}
\end{aligned} \tag{4}$$

3 Three Planes Intersect on the Centre

Planes p, q, r span points A, B, C on lines $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ paired with lines $\mathcal{P}_{qr}, \mathcal{Q}_{rp}, \mathcal{R}_{pq}$, respectively. A, B, C are on a, b, c , planes on origin O and normal to $\mathcal{P}, \mathcal{Q}, \mathcal{R}$. If any of these lines are on O then

its point contribution is just $O\{1 : 0 : 0 : 0\}$. Finding the first of three planes, $p\{P_0 : P_1 : P_2 : P_3\}$, will be done in detail. It is on point $A\{a_0 : a_1 : a_2 : a_3\}$, on plane $a\{A_0 : A_1 : A_2 : A_3\} = \{0 : p_{01} : p_{02} : p_{03}\}$, normal to \mathcal{P} . Planes q, r are obtained by similarity. Point A is obtained with the piercing point relation.

$$A = \mathcal{P} \cap a, \quad a_i = \sum_{j=0}^3 p_{ij} A_j, \quad p_{ij} = 0 \text{ if } i = j, \quad p_{ji} = -p_{ij}$$

$$\begin{aligned} a_0 &= p_{00}A_0 + p_{01}A_1 + p_{02}A_2 + p_{03}A_3 = p_{01}^2 + p_{02}^2 + p_{03}^2 \\ a_1 &= -p_{01}A_0 + p_{11}A_1 + p_{12}A_2 - p_{31}A_3 = p_{02}p_{12} - p_{03}p_{31} \\ a_2 &= -p_{02}A_0 - p_{12}A_1 + p_{22}A_2 + p_{23}A_3 = p_{03}p_{23} - p_{01}p_{12} \\ a_3 &= -p_{03}A_0 + p_{31}A_1 - p_{23}A_2 + p_{33}A_3 = p_{01}p_{31} - p_{02}p_{23} \end{aligned} \quad (5)$$

Similarly

$$\begin{aligned} b_0 &= q_{01}^2 + q_{02}^2 + q_{03}^2, \quad b_1 = q_{02}q_{12} - q_{03}q_{31}, \quad b_2 = q_{03}q_{23} - q_{01}q_{12}, \quad b_3 = p_{01}p_{31} - p_{02}p_{23} \\ c_0 &= r_{01}^2 + r_{02}^2 + r_{03}^2, \quad c_1 = r_{02}r_{12} - r_{03}r_{31}, \quad c_2 = r_{03}r_{23} - r_{01}r_{12}, \quad c_3 = r_{01}r_{31} - r_{02}r_{23} \end{aligned}$$

Planes p, q, r are obtained with the spanning plan relation. Note that P_{ij} are *axial* Plücker coordinates.

$$p = \mathcal{P}_{qr} \cap A, \quad P_i = \sum_{j=0}^3 P_{ij}^{qr} a_j, \quad P_{ij} = 0 \text{ if } i = j, \quad P_{ji} = -P_{ij}$$

$$\begin{aligned} P_0 &= P_{00}a_0 + P_{01}^{qr}a_1 + P_{02}^{qr}a_2 + P_{03}^{qr}a_3 \\ P_1 &= -P_{01}^{qr}a_0 + P_{11}a_1 + P_{12}a_2 - P_{31}a_3 \\ P_2 &= -P_{02}^{qr}a_0 - P_{12}a_1 + P_{22}a_2 + P_{23}a_3 \\ P_3 &= -P_{03}^{qr}a_0 + P_{31}a_1 - P_{23}a_2 + P_{33}a_3 \\ P_0 &= p_{23}^{qr}(p_{02}p_{12} - p_{03}p_{31}) + p_{31}^{qr}(p_{03}p_{23} - p_{01}p_{12}) + p_{12}^{qr}(p_{01}p_{31} - p_{02}p_{23}) \\ P_1 &= p_{03}(p_{03}p_{23} - p_{01}p_{12}) - p_{02}(p_{01}p_{31} - p_{02}p_{23}) - p_{23}^{qr}(p_{01}^2 + p_{02}^2 + p_{03}^2) \\ P_2 &= p_{01}(p_{01}p_{31} - p_{02}p_{23}) - p_{03}(p_{01}p_{12} - p_{03}p_{31}) - p_{31}^{qr}(p_{01}^2 + p_{02}^2 + p_{03}^2) \\ P_3 &= p_{02}(p_{01}p_{12} - p_{03}p_{31}) - p_{01}(p_{01}p_{31} - p_{02}p_{23}) - p_{12}^{qr}(p_{01}^2 + p_{02}^2 + p_{03}^2) \end{aligned} \quad (6)$$

By similarity one writes the coordinates of planes q and r .

$$\begin{aligned} Q_0 &= q_{23}^{rp}(q_{02}q_{12} - q_{03}q_{31}) + q_{31}^{rp}(q_{03}q_{23} - q_{01}q_{12}) + q_{12}^{rp}(q_{01}q_{31} - q_{02}q_{23}) \\ Q_1 &= q_{03}(q_{03}q_{23} - q_{01}q_{12}) - q_{02}(q_{01}q_{31} - q_{02}q_{23}) - q_{23}^{rp}(q_{01}^2 + q_{02}^2 + q_{03}^2) \\ Q_2 &= q_{01}(q_{01}q_{31} - q_{02}q_{23}) - q_{03}(q_{01}q_{12} - q_{03}q_{31}) - q_{31}^{rp}(q_{01}^2 + q_{02}^2 + q_{03}^2) \\ Q_3 &= q_{02}(q_{01}q_{12} - q_{03}q_{31}) - q_{01}(q_{01}q_{31} - q_{02}q_{23}) - q_{12}^{rp}(q_{01}^2 + q_{02}^2 + q_{03}^2) \end{aligned} \quad (7)$$

$$\begin{aligned} R_0 &= r_{23}^{pq}(r_{02}r_{12} - r_{03}r_{31}) + r_{31}^{pq}(r_{03}r_{23} - r_{01}r_{12}) + r_{12}^{pq}(r_{01}r_{31} - r_{02}r_{23}) \\ R_1 &= r_{03}(r_{03}r_{23} - r_{01}r_{12}) - r_{02}(r_{01}r_{31} - r_{02}r_{23}) - r_{23}^{pq}(r_{01}^2 + r_{02}^2 + r_{03}^2) \\ R_2 &= r_{01}(r_{01}r_{31} - r_{02}r_{23}) - r_{03}(r_{01}r_{12} - r_{03}r_{31}) - r_{31}^{pq}(r_{01}^2 + r_{02}^2 + r_{03}^2) \\ R_3 &= r_{02}(r_{01}r_{12} - r_{03}r_{31}) - r_{01}(r_{01}r_{31} - r_{02}r_{23}) - r_{12}^{pq}(r_{01}^2 + r_{02}^2 + r_{03}^2) \end{aligned} \quad (8)$$

Finally the centre point $M\{m_0 : m_1 : m_2 : m_3\}$ is defined by the point equation on planes p, q, r .

$$m_0 = \begin{vmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{vmatrix}, m_1 = \begin{vmatrix} P_2 & P_0 & P_3 \\ Q_2 & Q_0 & Q_3 \\ R_2 & R_0 & R_3 \end{vmatrix}, m_2 = \begin{vmatrix} P_0 & P_1 & P_3 \\ Q_0 & Q_1 & Q_3 \\ R_0 & R_1 & R_3 \end{vmatrix}, m_3 = \begin{vmatrix} P_1 & P_0 & P_2 \\ Q_1 & Q_0 & Q_2 \\ R_1 & R_0 & R_2 \end{vmatrix} \quad (9)$$