Constrained Fitting of a Rectangular Strut between Planes

(MECH289)CFRSP613.tex

January 30, 2006

1 Introduction

Consider the problem posed in Fig. 1 and solved there via descriptive geometric projections utilizing three principal views and first, second, third and two fourth auxiliary views. Cutting the strut to fit is based on information immediately measurable in the third and fourth auxiliaries. Overall length and *swing* angles at either end are available in the third while two fourths provide the *tilts* at each end. The eight corner points $ADGK$ on the floor and $BEJL$ on the sloping wall surface are obtained by projective reconstruction into the front and top views from the profile (right side), first and second auxiliary views.

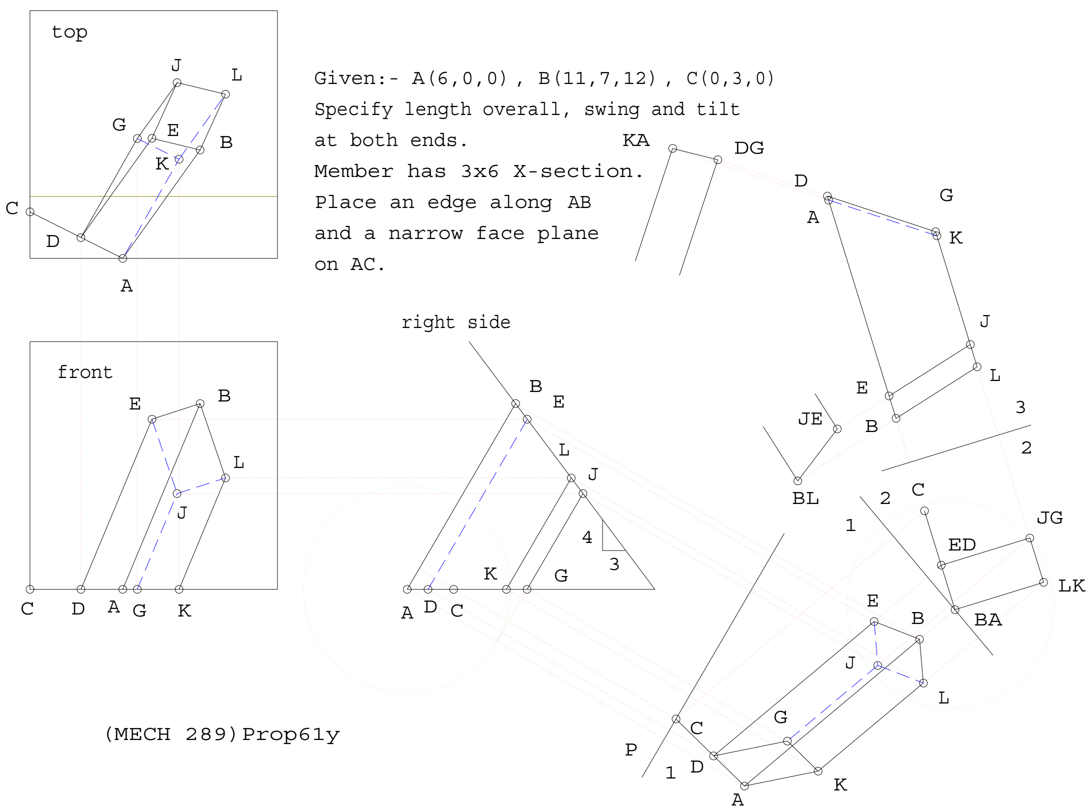

Figure 1: A 3×6 Strut between Planes at Arbitrary Angle; Two Points on an Edge, a Line on a Face

What is described here is the analytic geometry to extract the six plane faces and eight corner points that bound this strut. From this information the swing and tilt angles can be computed with the angle-between-line-and-plane and angle-between-two-planes methodology described in detail in one document **Geometric Construction Problem 1:- Molding Miter** and summarized in an other **Elementary Descriptive Geometry and Equivalent Linear Algebra** in sections **17** and **20**.

2 Points

Points are labeled in upper case. The following conventions are adopted.

$$A\{a_0 : a_1 : a_2 : a_3\} \equiv \mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

The same applies to the eight remaining points $BCDEGJKL$ on corner vertices. Note that four *homogeneous* point coordinates are presented separated by (:) to indicate that only *ratio* among them is relevant. If one sets $a_0 = 1$ or divides the other three by a_0 then the ordinary Cartesian coordinates of points in Euclidean space are obtained as given by the point position vector of A, *viz.*, $\mathbf{a}$. *E.g.*, $a_x = a_1/a_0$, *etc.*

2.1 A Point on Three Planes

Assume that coefficients of planes q, s and u are available. They intersect on point $J\{j_0 : j_1 : j_2 : j_3\}$. These coordinates may be obtained by expanding the following determinant on its top row minors. The plane $j\{J_0 : J_1 : J_2 : J_3\}$ is a dummy, variable plane on point J. Its coefficients are linearly dependent on the other three, given planes.

$$\begin{vmatrix} J_0 & J_1 & J_2 & J_3 \\ Q_0 & Q_1 & Q_2 & Q_3 \\ S_0 & S_1 & S_2 & S_3 \\ U_0 & U_1 & U_2 & U_3 \end{vmatrix} = \begin{vmatrix} Q_1 & Q_2 & Q_3 \\ S_1 & S_2 & S_3 \\ U_1 & U_2 & U_3 \end{vmatrix} J_0 - \begin{vmatrix} Q_0 & Q_2 & Q_3 \\ S_0 & S_2 & S_3 \\ U_0 & U_2 & U_3 \end{vmatrix} J_1 + \begin{vmatrix} Q_0 & Q_1 & Q_3 \\ S_0 & S_1 & S_3 \\ U_0 & U_1 & U_3 \end{vmatrix} J_2 - \begin{vmatrix} Q_0 & Q_1 & Q_2 \\ S_0 & S_1 & S_2 \\ U_0 & U_1 & U_2 \end{vmatrix} J_3 = 0$$

This is like a plane equation where coefficients and variables have traded places.

$$j_0 J_0 + j_1 J_1 + j_2 J_2 + j_3 J_3 = 0$$

3 Planes

Planes are labeled in lower case. The following conventions are adopted.

$$\begin{gathered} t(ABED) \equiv t\{T_0 : T_1 : T_2 : T_3\} \\ u(GJLK) \equiv u\{U_0 : U_1 : U_2 : U_3\} \\ r(BAKL) \equiv r\{R_0 : R_1 : R_2 : R_3\} \\ s(DEJG) \equiv s\{S_0 : S_1 : S_2 : S_3\} \\ p(ADGK) \equiv p\{P_0 : P_1 : P_2 : P_3\} \\ q(BLJE) \equiv q\{Q_0 : Q_1 : Q_2 : Q_3\} \end{gathered}$$

Note that, *e.g.*, plane s is on the four corner points ($DEJG$), with right-hand circulation sequence in the outward sense, while $s\{S_0 : S_1 : S_2 : S_3\}$ represents its linear equation coefficients in the sequence $\{\text{constant} : x : y : z\}$. These coefficients are also called *homogeneous coordinates* of the plane.

3.1 Plane t

Since ABC are given, the outward normal to t is given by the following vector cross-product.

$$\mathbf{b} - \mathbf{a} \times \mathbf{c} - \mathbf{a} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

Then with A, or either of the two other points B or C that are all on t, the plane equation is used to find T_0.

$$T_0 = -(T_1 a_x + T_2 a_y + T_3 a_z)$$

3.2 Planes p and q

These are the cut end faces. By inspection

$$p\{0:0:0:P_3\}$$

If one desires an outward normal $P_3 = -1$ will serve nicely. Since the slope of plane q is given as $3:4$ and has no slope in the x-direction

$$q\{Q_0:0:Q_2:Q_3\}$$

the outward normal can be taken as $Q_2 = 4,\ \ Q_3 = 3$. Then notice that point B is on q so

$$Q_0 = -(Q_2 b_y + Q_3 b_z)$$

3.3 Plane r

The outward normal of r can be calculated as the cross-product of a vector in direction AB pre-multiplying the normal vector of t.

$$(\mathbf{b}-\mathbf{a}) \times \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

Since A and B are on r as well as t, T_0 can be given by

$$R_0 = -(T_1 a_x + T_2 a_y + T_3 a_z) = -(R_1 a_x + R_2 a_y + R_3 a_z)$$

3.4 Planes s and u

The outward normal of s is just that of r, negated.

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = -\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

A point on s is needed to get S_0 so just add 3 times the unit vector in the direction AC to get point D. The narrow width of the rectangular cross-section is of dimension 3 units.

$$\mathbf{d} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \mathbf{a} + 3\frac{\mathbf{c}-\mathbf{a}}{\|\mathbf{c}-\mathbf{a}\|}$$

and

$$S_0 = -(S_1 d_x + S_2 d_y + S_3 d_z)$$

The normal of plane u is in the opposite sense to that of t.

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

To get a point on u we cannot go for, say, K or L however we *can* go 6 units from, say, A in the direction of the outward normal to u. Let us call this point K'.

$$\begin{bmatrix} k'_x \\ k'_y \\ k'_z \end{bmatrix} = \mathbf{a} + \frac{6}{\sqrt{U_1^2 + U_2^2 + U_3^2}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

Finally U_0 can be calculated.

$$U_0 = -(U_1 k'_x + U_2 k'_y + U_3 k'_z)$$

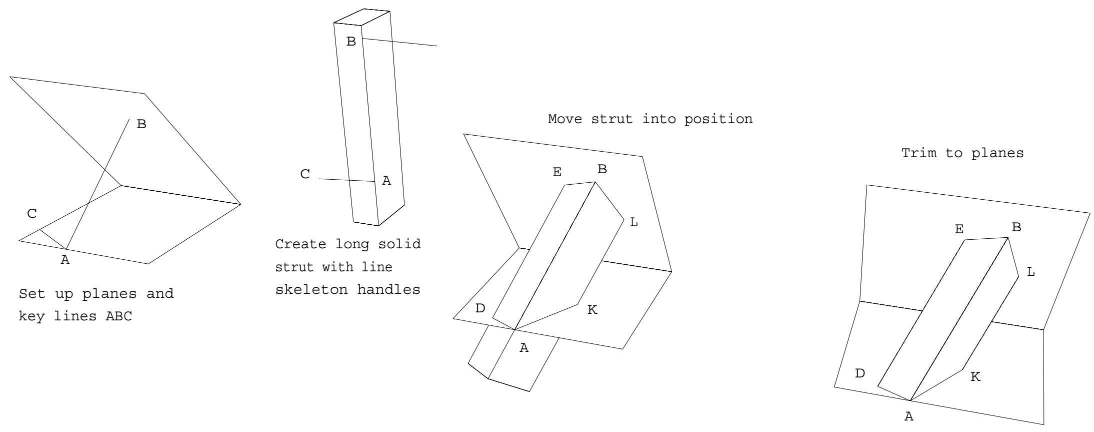

Figure 2: Designing the Strut as a Solid Model

4 A 3D CAD Approach

Now examine Fig. 2. In the leftmost picture the two planes p and q have been constructed together with the points ABC and the line segments joining them. These three points will be the “handles” whereby the the “piece of lumber” will be transferred to the desired position. Building the two planes and the 3×6 rectangular parallelepiped is easy but to “paste” ABC on the appropriate face of the 3×6 one needs the distances AB and AC and the angle α at A.

$$\|\mathbf{b}-\mathbf{a}\| = \sqrt{5^2+7^2+12^2} = 14.7648, \quad \|\mathbf{c}-\mathbf{a}\| = \sqrt{6^2+3^2} = 6.7082$$

$$\cos\alpha = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})}{\|\mathbf{b}-\mathbf{a}\| \; \|\mathbf{c}-\mathbf{a}\|} = \frac{\begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix}}{\sqrt{25+49+144}\sqrt{36+9}} = -0.090867$$

(MECH289)CFRSP613.tex