MECH 576
Geometry in Mechanics

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A Cylinder of Revolution on Five Points

1 Introduction

In January 2006 a fascinating seminar was conducted by representatives of *FARO Technologies, Inc.*, a hi-tech Florida based company, on their state-of-the art, six revolute jointed serial robot arm that can be equipped with various metrological end effectors ranging from a simple miniature touch-sphere to a raster scanning laser rangefinder that can capture thousands of points on a spatial surface and resolve their location to within a sphere of 25μ radius as the user manually moves the end effector to scan the part. One example that was demonstrated was the encoding and display of a cylindrical hole in a manufactured part by touching various points on its surface. The device is capable of incorporating redundant points, *i.e.*, much greater than the minimum to determine a cylinder, and statistically establishing a consensus that may enhance the accuracy of the solid model obtained far beyond that achieved with a minimum number of touches. “How many represent a minimum number?” was asked. “Six.” was the reply. This established a reason for this article. Representatives of a corporation of some 600 employees, successfully selling a precision instrument for over U$100000, seemed unaware of essential geometric principles upon which their product is based. Using descriptive and analytical geometry, we outline and test a robust and efficient algorithm to unambiguously identify cylinders of revolution on five points by finding their axial direction. The results are not new since there exists a considerable body of modern pertinent literature, *e.g.*, by Devilliers, *et al* [1]. It is interesting to note that this work in English makes no reference to earlier research in German. We take this opportunity to expose what is believed to be a somewhat novel and elegant approach based on simple engineering vector algebra.

2 A Key Equation and a Solution

Professor Manfred Husty of the University of Innsbruck, together with Dr. Hans-Peter Schröcker, devised and solved a system of four simultaneous equations based on earlier work by Professor Hermann Schaal who studied the distance relation between points on a cylinder of revolution [2] and the one parameter system of such cylinders specified to lie on five given points [3]. These equations are of the following form.

$$(\mathbf{x} \times \mathbf{a})^2 - 2\mathbf{a}^2(\mathbf{x} \cdot \mathbf{f}) = 0 \tag{1}$$

where $\mathbf{x}$ indicates the position vector from an origin, O, chosen to be on one of the given points, to each of the other four. $\mathbf{a}$ is any unknown vector in the direction of the cylinder’s axis and all its generators. $\mathbf{f}$ is the vector from O to the axis and normal to $\mathbf{a}$. Derivation of Eq. 1 is detailed in the appendix, section **4**. The two additional conditions, necessary to obtain the six unknowns, *i.e.*, elements of $\mathbf{a}$ and $\mathbf{f}$, are

$$\mathbf{a}\cdot\mathbf{f}=0,\ \ \mathbf{a}^2=1 \tag{2}$$

The five given points are O, P, Q, R, S. There is no loss in generality if O is taken as the Cartesian origin, P is on the x-axis and Q is on the plane $z=0$. This gives

$$\mathbf{o}=\begin{bmatrix}0\\0\\0\end{bmatrix},\ \mathbf{p}=\begin{bmatrix}p_1\\0\\0\end{bmatrix},\ \mathbf{q}=\begin{bmatrix}q_1\\q_2\\0\end{bmatrix},\ \mathbf{r}=\begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix},\ \mathbf{s}=\begin{bmatrix}s_1\\s_2\\s_3\end{bmatrix} \tag{3}$$

Eq. 1 and the first of Eq. 2 produce five equations in the six unknowns

$$\mathbf{a}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix},\ \mathbf{f}=\begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix} \tag{4}$$

$$\begin{aligned}
a_1f_1+a_2f_2+a_3f_3&=0\\
p_1^2a_3^2+p_1a_2^2-2p_1(a_1^2+a_2^2+a_3^2)f_1&=0\\
q_2^2a_3^2+q_1^2a_3^2+(q_1a_2-q_2a_1)^2-2(a_1^2+a_2^2+a_3^2)(q_1f_1+q_2f_2)&=0\\
(r_2a_3-r_3a_2)^2+(r_3a_1-r_1a_3)^2+(r_1a_2-r_2a_1)^2-2(a_1^2+a_2^2+a_3^2)(r_1f_1+r_2f_2+r_3f_3)&=0\\
(s_2a_3-s_3a_2)^2+(s_3a_1-s_1a_3)^2+(s_1a_2-s_2a_1)^2-2(a_1^2+a_2^2+a_3^2)(s_1f_1+s_2f_2+s_3f_3)&=0
\end{aligned} \tag{5}$$

Elimination of f_1, f_2, f_3 from the system Eq. 5 leaves two equations, Eq. 6, in a_1, a_2, a_3.

$$\begin{aligned}
b_1a_1a_2a_3+b_2a_1a_2^2+b_3a_1a_3^2+b_4a_3^3+b_5a_2a_3^2+b_6a_1^2a_2+b_7a_2^3+b_8a_1^2a_3+b_9a_2^2a_3&=0\\
c_1a_1^2+c_2a_2^2+c_3a_3^2+c_4a_2a_3+c_5a_1a_3+c_6a_1a_2&=0
\end{aligned} \tag{6}$$

Rather than taking $\mathbf{a}$ to be a unit vector, as implied by the second of Eq. 2, we set $a_3=1$ instead. Note that the constraints imposed by Eq. 5 are independent of the magnitude and sense of $\mathbf{a}$. If a_3 turns out to be $a_3=0$, due to the disposition of the five given points, one may set either $a_2=1$ or $a_1=1$. Since a_1 does not appear as a_1^3 we choose to eliminate a_1 in Eq. 6.

$$\begin{aligned}
(b_6a_2+b_8)a_1^2+(b_2a_2^2+b_1a_2+b_3)a_1+(b_7a_2^3+b_9a_2^2+b_5a_2+b_4)&=0\\
c_1a_1^2+(c_6a_2+c_5)a_1+(c_2a_2^2+c_4a_2+c_3)&=0
\end{aligned} \tag{7}$$

Upon eliminating a_1 a sestic univariate is obtained.

$$D_1a_2^6+D_2a_2^5+D_3a_2^4+D_4a_2^3+D_5a_2^2+D_6a_2+D_7=0 \tag{8}$$

First we define the seven coefficients $D_i,\ \ i=1,\ \ldots,\ \ 7$ in terms of the already compressed coefficients that engender the given point coordinates of O, P, Q, R, S, *i.e.*, $b_j,\ \ j=1,\ \ldots,\ \ 9$ and $c_k,\ \ k=1,\ \ldots,\ \ 6$.

$$\begin{aligned}
D_1=d_1d_2-d_3^2,\ \ D_2=d_4d_2+d_1d_5-2d_8d_3,\ \ D_3=d_9d_2+d_4d_5+d_1d_6-2d_7d_3-d_8^2\\
D_4=d_9d_5+d_4d_6+d_1d_{10}-2d_{11}d_3-2d_7d_8,\ \ D_5=d_9d_6+d_4d_{10}+d_1d_{12}-2d_{11}d_8-d_7^2\\
D_6=d_9d_{10}+d_4d_{12}-2d_{11}d_7,\ \ D_7=d_9d_{12}-d_{11}^2
\end{aligned} \tag{9}$$

where

$$\begin{gathered}
d_1 = c_1 b_2 - b_6 c_6, \;\; d_2 = c_6 b_7 - b_2 c_2, \;\; d_3 = c_1 b_7 - b_6 c_2, \;\; d_4 = c_1 b_1 - b_8 c_6 - b_6 c_5 \\
d_5 = c_5 b_7 + c_6 b_9 - b_1 c_1 - b_2 c_4, \;\; d_6 = c_5 b_9 + c_6 b_5 - b_3 c_2 - b_1 c_4 - b_2 c_3, \;\; d_7 = c_1 b_5 - b_8 c_4 - b_6 c_3 \\
d_8 = c_1 b_9 - b_8 c_2 - b_6 c_4, \;\; d_9 = c_1 b_3 - b_8 c_5, \;\; d_{10} = c_5 b_5 + c_6 b_4 - b_3 c_4 - b_1 c_3 \\
d_{11} = c_1 b_4 - b_8 c_3, \;\; d_{12} = c_5 b_4 - b_3 c_3 \quad (10)
\end{gathered}$$

These 15 additional coefficients, though not trivial, are nevertheless neither too extensive to write out.

$$\begin{gathered}
b_1 = 2q_2 r_2 (q_1 - r_1), \;\; b_2 = q_2 r_3 (p_1 - 2q_1), \;\; b_3 = q_2 r_3 (p_1 - 2r_1) \\
b_4 = q_2 [r_1 (r_1 - p_1) + r_2 (r_2 - q_2)] + q_1 r_2 (p_1 - q_1) \\
b_5 = r_3 [q_2 (q_2 - 2r_2) + q_1 (q_1 - p_1)], \;\; b_6 = q_2^2 r_3 \\
b_7 = q_1 r_3 (q_1 - p_1), \;\; b_8 = q_2 [r_2 (r_2 - q_2) + r_3^2], \;\; b_9 = q_2 [r_1 (r_1 - p_1) + r_3^2] + q_1 r_2 (p_1 - q_1) \\
c_1 = q_2 [r_2 s_3 (r_2 - q_2) + r_3 s_2 (q_2 - s_2) + r_3 s_3 (r_3 - s_3)] \\
c_2 = q_2 r_3 s_3 (r_3 - s_3) + q_1^2 (r_3 s_2 - r_2 s_3) + p_1 [q_2 (r_3 s_1 - r_1 s_3) + q_1 (r_2 s_3 - r_3 s_2)] + q_2 (r_1^2 s_3 - r_3 s_1^2) \\
c_3 = q_2 [p_1 (s_1 r_3 - s_3 r_1) - r_3 (s_1^2 + s_2^2) + s_3 (r_1^2 + r_2^2)] + (r_2 s_3 - r_3 s_2)[q_1 (p_1 - q_1) - q_2^2] \\
c_4 = 2q_2 r_3 s_3 (s_2 - r_2), \;\; c_5 = 2q_2 r_3 s_3 (s_1 - r_1), \;\; c_6 = 2q_2 [r_3 s_2 (s_1 - q_1) + r_2 s_3 (q_1 - r_1)] \quad (11)
\end{gathered}$$

Once values of a_2 have been determined with Eq. 8 a linear equation in a_1 can be produced by eliminating a_2^2 between Eqs. 7, thus.

$$[c_1(b_2 a_2^2 + b_1 a_2 + b_3) - (b_6 a_2 + b_8)(c_6 a_2 + c_5)] a_1 + [c_1(b_7 a_2^3 + b_9 a_2^2 + b_5 a_2 + b_4) - (b_6 a_2 + b_8)(c_2 a_2^2 + c_4 a_2 + c_3)] = 0 \quad (12)$$

2.1 A Numerical Example

The five given points

$$O(0,0,0), \;\; P(3,0,0), \;\; Q(2,2,0), \;\; R(0,2,4), \;\; S(2,0,3)$$

were coded into Eqs. 11, 10 and 9, in that order, and solved for a_2 with Eq. 8. The two real roots were used in Eq. 12 to get the corresponding values of a_1. Of course $a_3 = 1$ in all cases.

a_3	a_2	a_1
1	0.3876994216	0.01221017201
1	$2.079377226 + 1.357288502i$	$\cdots$
1	$-1.32199443 + 0.3554442093i$	$\cdots$
1	-0.01630698077	-0.3397443430
1	$-1.32199443 - 0.3554442093i$	$\cdots$
1	$2.079377226 - 1.357288502i$	$\cdots$

There are a number of ways to find the two cylinders represented by the first and fourth axial direction numbers, the only real ones, tabulated above. *E.g.*, one may find four additional points on each cylinder by parameterizing lines on four of the given points, say, P, Q, R, S, using a vector on the point $A_\omega\{0 : a_1 : a_2 : a_3\}$ multiplied by the parameter t and finding the point intersections P_a, Q_a, R_a, S_a on the plane $a\{A_0 : A_1 : A_2 : A_3\} = \{0 : a_1 : a_2 : a_3\}$ on O and normal to these lines. Pertinent equations to define, say, P_a and find t are as follows.

$$P_a :\ \mathbf{p}_a = \mathbf{p} + \mathbf{a}_\omega t, \quad \begin{bmatrix} p_{A1} \\ p_{A2} \\ p_{A3} \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} a_1 t \\ a_2 t \\ a_3 t \end{bmatrix}$$

$$(p_1 + a_1 t)a_1 + (p_2 + a_2 t)a_2 + (p_3 + a_3 t)a_3 = 0$$

Then these nine points may be used to find the ten quadric coefficients on the cylinder by Grassmannian expansion of the 10×10 determinant of the singular matrix of the ten quadratic variable point forms in the implicit equation of the quadric. These two implicit equations are plotted to show the two real cylinders in Fig. 1.

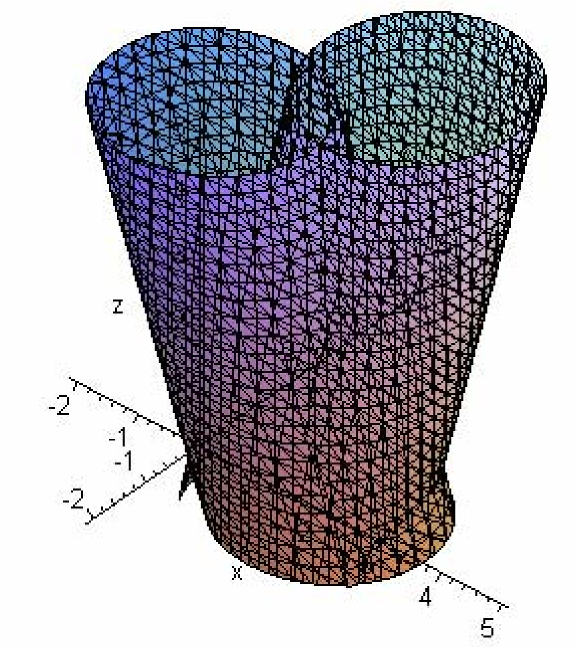

Figure 1: Two Real Solutions on Given Example with Five Specific Points

But are these two quadrics really and truly cylinders of revolution? What one sees in Fig 2 is a descriptive geometric construction that is equivalent to projecting all five points, including O, onto plane a, finding a circular section centre on the intersection of right bisectors of chords, say, $O_A P_A$ and $Q_A R_A$, and establishing that the distances from such a centre to all five point projections are identical.

This constructive verification, that involves only an end view of any vector $\mathbf{a}[a_1 \;\; a_2 \;\; a_3]^T$, seems to be more convenient than algebraically finding four planes and a line to establish the condition of equidistance.

2.2 Six Solutions

It is not hard to imagine a cube of unit edge length where $O(0,0,0), P(1,0,0), Q(1,1,0), R(0,0,1), S(1,1,1)$ are vertices. Three cylinders of revolution circumscribe the three pairs of parallel face squares.

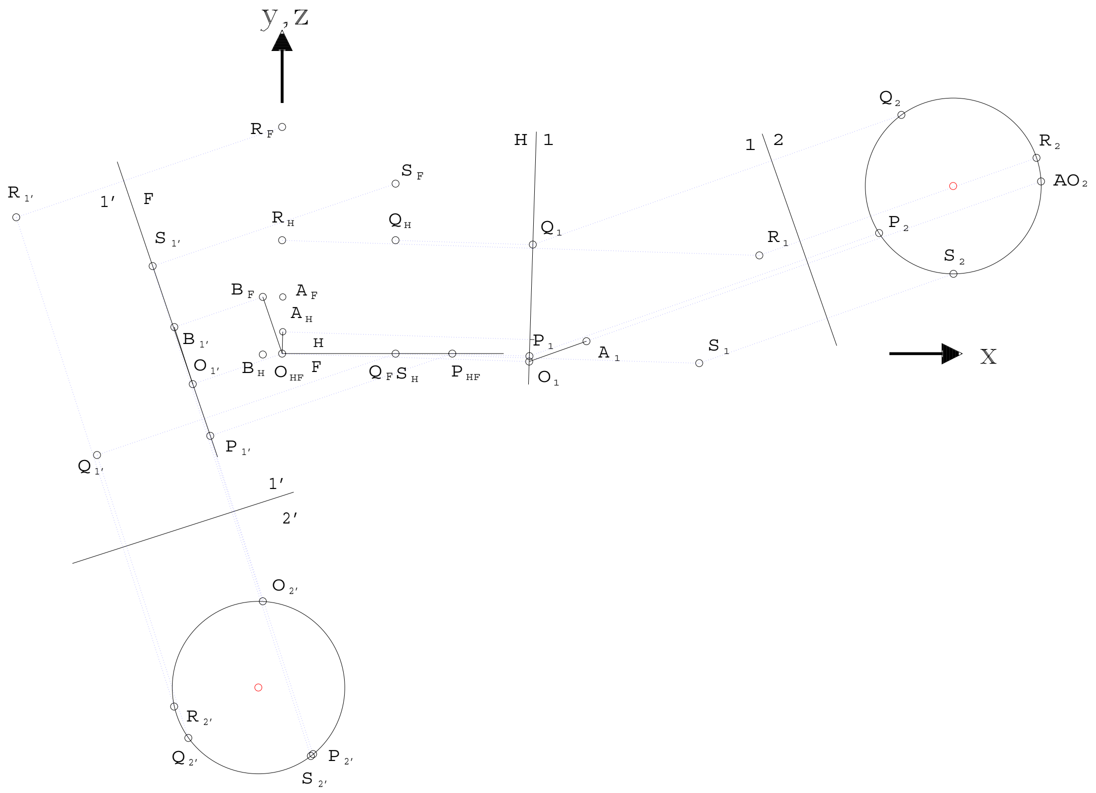

Figure 2: The Solutions Are Verified with Circular Sections Obtained by Projection in Computed Axial Directions

A fourth may be constructed perpendicular to a plane normal to parallel face diagonals OQ and RS. Another cylinder whose generators are parallel to the cube's space diagonal that is coplanar with the one on P also contains all five points which appear on the vertices of a regular hexagon when the cylinder is viewed in a projection normal to cylinder axis. However there does not appear to be a sixth cylinder. There is a double solution. However Dr. A. Gfrerrer at TU-Graz has shown that six distinct real solutions may indeed exist, *e.g.*, given the point array $O(0,0,0), P(5,0,0), Q(0,5,0), R(0,0,5), S(5,4,4)$.

3 Conclusion

Although this particular problem may appear to have been thoroughly and sufficiently worked over, nevertheless there remain related issues as yet unsolved. These involve the definition of other special quadrics that may be specified by less than nine points, *e.g.*, surfaces of revolution, and the incorporation of over-determined point sets such as may be encountered in inspection and metrology based on digitized optical data. For instance, the *FARO* robot introduced in section **1** can measure cylindrical shafts and holes using many more than five surface points however no references that deal with the nature and efficiency of the procedure, used to process redundant data, have been discovered.

4 Appendix

Fig. 3 shows an axial view of the cylinder. O is one of the five given points. There is no loss in generality to take it as the Cartesian origin. A is any point on the generator on O while X represents any one of the other four given points on the cylinder. $\mathbf{x}$ spans OX, $\mathbf{a}$ spans OA and is parallel to the axis while $\mathbf{f}$ spans from O to the axis and is normal to it. A constructive solution follows. Note that $\mathbf{x}\times\mathbf{a}/|\mathbf{a}|$ and $(\mathbf{x}\cdot\mathbf{f})\mathbf{f}/\mathbf{f^2}$ represent a vector whose *length* is that of the projection of $\mathbf{x}$ normal to axial direction parallel to $\mathbf{a}$ and vector, along $\mathbf{f}$, of length equal to the projection of $\mathbf{x}$ on $\mathbf{f}$, respectively. A purely geometric construction to obtain the circle whose diameter is $2\mathbf{x}\cdot\mathbf{f}/|\mathbf{f}|$ is shown in Fig. 3. Note how the length of $\mathbf{x}\times\mathbf{a}/|\mathbf{a}|$ is transferred to the vertical side of the right triangle on base OX and, with an arc centred on X, cuts the diameter $2|\mathbf{f}|$ of the circular section of the cylinder.

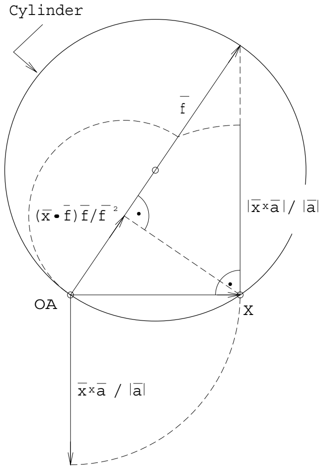

Figure 3: The Cylinder and Vectors Pertaining to Eq. 1

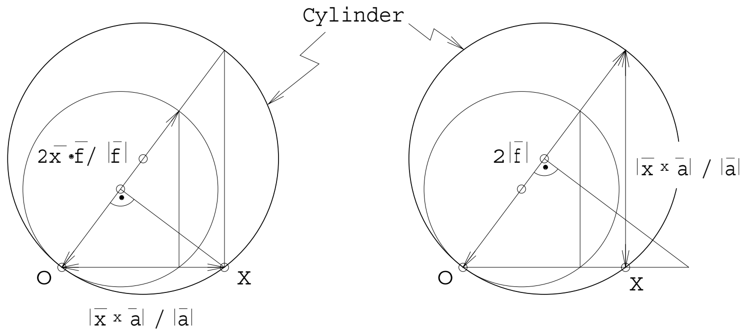

Figure 4: Scaling via Similar Triangles

Now Fig. 4 shows, via similar triangles, the proportionality between the radius $|\mathbf{f}|$ of the cylinder section and the smaller, construction circle radius to be

$\frac{\mathbf{x}\cdot\mathbf{f}/|\mathbf{f}|}{|\mathbf{f}|} = \frac{|\mathbf{x}\times\mathbf{a}|}{|\mathbf{a}||\mathbf{f}|}$ Therefore one may write $\frac{2\mathbf{x}\cdot\mathbf{f}}{|\mathbf{f}|} = \frac{|\mathbf{x}\times\mathbf{a}||\mathbf{x}\times\mathbf{a}|}{|\mathbf{a}||\mathbf{a}||\mathbf{f}|}$ which is seen to be Eq. 1

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References

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