DOUBLE 3D PENDULUM

Control Problem

Goal in this experiment:

Partial, semi-global stabilization to the unstable upright, inverted equilibrium.

- Partial stabilization is defined as stabilization to a subspace in the phase space of the system that excludes the rotational motion of the lower body around the vertical z-axis. Spinning of the pendulum around the vertical axis is permitted, albeit not observed in this control design experiment.

- Semi-global stabilization is defined as stabilization from a maximally large set of initial positions of the pendulum.

Assumptions:

- There are no constraints on the available actuation power;

- The ground, or floor, is not a motion obstacle, i.e. the pendulum is free to move about its "foot".

Characteristics & Difficulty of the Problem:

- The system has several relative equilibria with different stability characteristic; see:

  Marsden, J. E., Scheurle, "Lagrangian Reduction and the Double Spherical Pendulum"; Zeitschrift Fur Angewandte Mathematik Und Physik - ZAMP , vol. 44, no. 1, pp. 17-43, 1993.

- Due to assumed loss of friction in the joints, the uncontrolled system exhibits sustained chaotic behavior;

- The system is under-actuated and is represented by 10 states of which 8 need to be stabilized by the action of only two control variables. Linearization of the system around the upright, inverted equilibrium is partially stabilizable, but linear control has negligibly small basin of attraction.

- Due to the presence of gravity, the system can be considered as a non-holonomic system of order two with acceleration constraints that are fully non-integrable – the constraints cannot be integrated to equivalent velocity constraints.

- The system cannot be globally partially stabilized due to its non-holomomy and absence of a global atlas on the configuration space $SO(3) \times SO(3)$.

- From a structural point of view, the 3D inverted pendulum stabilization problem is very different from the corresponding planar 2D inverted pendulum stabilization problem as the 3D system does not have kinetic symmetries. The kinetic energy of the system is not cyclic with respect to the un-actuated variables. For this reason planar approaches cannot be applied directly; see:

  Grizzle, J. W., Chevallereau, C, Moog, C. H., "Nonlinear control of mechanical systems with an unactuated cyclic variable", IEEE Transactions on Automatic Control, vol. 50 (5), pp. 559-576, 2005.

- Stabilization based on symmetry reduction techniques, coupled with energy shaping and passivity based control remains a challenge of the future if only because it leads to prohibitively complex calculations; the feasibility and outcome of which is yet to be determined. Although a conceptual control design strategy for stabilization of a similar 3D pendulum (with a spherical fully actuated joint at the "hip") to one of its hanging relative equilibria (referred to as the "cowboy equilibrium") has been proposed, no end results are reported; see:

  Jalnapurkar, S. M., Marsden, J. E., "Stabilization of relative equilibria", IEEE Transactions on Aut. Control, vol. 45 (8), pp. 1483-1491, 2000.

A Few Hints
Towards a Computationally Feasible
Stabilizing Control Design Approach

- The $x, y-$projections l_x, l_y of the angular momentum of the pendulum w.r.t. its "foot", when adopted as the system output, has vector relative degree $(3, 3)$ with respect to the $x, y-$components of the reaction force, f_x, f_y at the foot; see the Appendix at the end of this file.

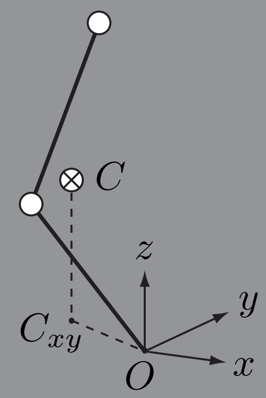
- The output assignment and its globally well defined vector relative degree w.r.t. the fictitious inputs f_x, f_y permit for linear control of 6 degrees of freedom in the pendulum system. The linear control is global provided that no singularity is encountered in inversion of the function that relates the reaction forces at the foot to the input torques at the "hip". It can be verified that the zero dynamics of the controlled system is stable, but not asymptotically stable.

- Asymptotic partial stabilization of the zero dynamics must be achieved by way of the action of the vertical component, f_z, of the reaction force at the foot which cannot be accessed by the control torques in an instantaneous and direct manner. This task can be accomplished by a nonlinear control "correction" aiming to balance the energies in the system with the work performed by the vertical component of the reaction force f_z , with the outcome shown in simulations. Analysis is complex.

APPENDIX

Calculation of the relative degree of the system output defined to be the moment of momentum of the pendulum w.r.t. its "foot".

Notation



- m – total mass of the pendulum.
- $\boldsymbol{c}$ – center of mass of the pendulum ;
  $\boldsymbol{c}_{xy} = [c_x, c_y]$ – projection of $\boldsymbol{c}$.
- $\boldsymbol{v}$ – velocity of $\boldsymbol{c}$;
  $\boldsymbol{v}_{xy} = [v_x, v_y]$ – projection of $\boldsymbol{v}$.
- $\boldsymbol{h} = m\boldsymbol{v}$ – linear momentum of the pendulum;
  $\boldsymbol{h}_{xy} = [h_x, h_y]$ – projection of $\boldsymbol{h}$.
- $\boldsymbol{l} = \boldsymbol{c} \times \boldsymbol{h}$ – angular momentum (moment of momentum) of the pendulum w.r.t. the "foot"; $\boldsymbol{l}_x y = [l_x, l_y]$ – projection of $\boldsymbol{l}$.
- $\boldsymbol{f}_g = m\boldsymbol{g} = m[0, 0, -g]^\top$ – gravitational force acting on the center of mass.
- $\boldsymbol{\tau}_g = \boldsymbol{c}_{xy} \times \boldsymbol{f}_g$ – gravitational torque around the "foot".
- $\boldsymbol{f}$ - reaction force at the foot; $\boldsymbol{f}_{xy} = [f_x, f_y]$ – projection of $\boldsymbol{f}$.
- $\boldsymbol{\tau}$ – actuated torque at the hip .

Kinematic and dynamic dependencies:

$$\boldsymbol{v}_{xy} = \dot{\boldsymbol{c}}_{xy} = \frac{1}{m}\boldsymbol{h}_{xy} \tag{1}$$

$$\dot{\boldsymbol{h}}_{xy} = \boldsymbol{f}_{xy} \tag{2}$$

$$\dot{\boldsymbol{l}}_{xy} = \boldsymbol{\tau}_g = mg\,\boldsymbol{K}\boldsymbol{c}_{xy} \tag{3}$$

with $\boldsymbol{\tau}_g = \boldsymbol{c}_{xy} \times \boldsymbol{f}_g = [c_x, c_y, 0]^\top \times [0, 0, -mg]^\top = mg \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_x \\ c_y \end{pmatrix}$

and hence $\boldsymbol{K} \triangleq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Differentiating (3) gives

$$\begin{aligned} \ddot{\boldsymbol{l}}_{xy} &= mg\,\boldsymbol{K}\boldsymbol{v}_{xy} = g\boldsymbol{K}\boldsymbol{h}_{xy} \\ \dddot{\boldsymbol{l}}_{xy} &= g\,\boldsymbol{K}\boldsymbol{f}_{xy} \end{aligned} \tag{4}$$

If $\boldsymbol{l}_{xy}$ is defined the vector output of the system then it has vector relative degree $(3,3)$ with respect to $\boldsymbol{f}_{xy}$ and hence also with respect to the torque input $\boldsymbol{\tau}_g$ that directly affects it.