Sines and cosines: the basis of the Fourier transform

We next consider an important class of functions, namely sinusoids, and their special behaviour under convolution. Take a cosine function with \( k \) cycles from \( x = 0 \) to \( x = N \), where \( k \) is an integer,

\[
\cos\left(\frac{2\pi k}{N} x\right).
\]

Note that this cosine function has the same value at \( x = N \) as at \( x = 0 \).

Suppose we were to convolve the cosine with a function \( h(x) \).

\[
h(x) \ast \cos\left(\frac{2\pi}{N} x\right) = \sum_{x'} \cos\left(\frac{2\pi}{N} (x - x')\right) h(x')
\]

Recalling that

\[
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

we see that the right hand side is just a sum of sine and cosine functions with variable \( x \) and constant frequency \( k \), and so it can be written

\[
h(x) \ast \cos\left(\frac{2\pi}{N} x\right) = a \cos\left(\frac{2\pi}{N} kx\right) - b \sin\left(\frac{2\pi}{N} kx\right)
\]

for some \( a \) and \( b \) which depend on \( k \) and on the function \( h(x) \). Specifically,

\[
a = \sum_{x'} h(x') \cos\left(\frac{2\pi}{N} x'\right)
\]

\[
b = \sum_{x'} h(x') \sin\left(\frac{2\pi}{N} x'\right)
\]

which are just the inner products of the \( N - D \) vectors \( h(\cdot) \) with a cosine or sine of frequency \( k \), respectively.

Define angle \( \phi \) such that

\[
(\cos \phi, \sin \phi) = \frac{1}{\sqrt{a^2 + b^2}} (a, b).
\]

Then

\[
h \ast \cos\left(\frac{2\pi}{N} x\right) = \sqrt{a^2 + b^2} \left(\cos(\phi) \cos\left(\frac{2\pi}{N} kx\right) + \sin(\phi) \sin\left(\frac{2\pi}{N} kx\right)\right)
\]

\[
= \sqrt{a^2 + b^2} \cos\left(\frac{2\pi}{N} kx - \phi\right)
\]

\( \sqrt{a^2 + b^2} \) is called the amplitude and \( \phi \) is called the phase. The amplitude and phase shift depend on frequency \( k \) and on the function \( h(\cdot) \).

Quick summary: convolving a cosine with an arbitrary function \( h(x) \) gives you back a cosine of the same frequency \( k \), though possibly phase shifted in position \( x \). Exactly the same argument can be made for a sine function.

Our next task is to show how to represent any function \( I(x) \) as a sum of sines and cosines. Why would we want to do this? We just saw that sines and cosines behave very nicely under convolution,
namely convolving a sine or cosine with any function $h(x)$ always yields a (possibly shifted) sine with the same frequency, but with possible different amplitudes. This, along with the distributive law, implies that convolving $I(x)$ with any function $h(x)$ would give the same result as if we were to decompose $I(x)$ into a sum of sines and cosines, convolve each of these with $h(x)$, and then add the results together. This result is the essence of the convolution theorem which we will cover next lecture.

Before we show how to represent any function as a sum of sines and cosines, let’s review some basics of complex numbers.

**Complex numbers (review)**

To decompose functions into sines and cosines we are going to use complex variables. A key trick is *Euler’s equation*:

$$ e^{i\theta} = \cos \theta + i \sin \theta $$

where $i^2 = -1$. The geometric picture you should have is that $e^{i\theta}$ represents a point on the unit circle in the complex plane.

Here are some examples:

$$ e^{i 0} = 1, \quad e^{i \pi/2} = i, \quad e^{i \pi} = -1, \quad e^{i \pi/4} = \frac{1}{\sqrt{2}} (1 + i), \quad e^{i 2\pi n} = 1 \text{ for any integer } n $$

To understand what Euler’s equation means, you need to remember what $e^x$ means for any $x$, namely

$$ e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^j}{j!} \cdots $$

When $x$ is an integer, you don’t need to interpret $e^x$ in this complicated way. But if $x$ is more general e.g. an irrational number, or a square matrix, or complex number, then you should interpret it in this way.
Now where does Euler’s equation come from? Plugging in $i\theta$ for $x$ in the series above, we get

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

Euler’s equation comes from the fact that the Taylor series expansion of $\sin \theta$ and $\cos \theta$ about $\theta = 0$ are

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

and

$$\cos \theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots$$

Euler’s equation is useful because we can apply the usual rules of multiplication, where we add exponents. (It can be shown that this follows from the power series representation of $e^x$).

For example, Euler’s equation provides us with familiar trigonometric identities. Consider

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1 + \theta_2)} = \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)$$

But

$$e^{i\theta_1}e^{i\theta_2} = (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)$$

$$= (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)$$

which gives you familiar identities for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$.

A few other familiar definitions: For any complex number, $c = a + bi$, the complex conjugate is defined $\overline{c} = a - bi$. The complex conjugate has the property that

$$c \overline{c} = |c|^2 = a^2 + b^2$$

In particular, $e^{-i\theta}$ is the complex conjugate of $e^{i\theta}$ and

$$e^{i\theta}e^{-i\theta} = 1.$$

The complex conjugate of $c$ should not be confused with the inverse of $c$, namely the complex number $c^{-1}$ which satisfies $cc^{-1} = 1$,

$$c^{-1} = \frac{1}{|c|} \overline{c}.$$

Finally, recall that the inner product of two $N$-dimensional vectors of complex numbers is defined:

$$(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{N} \overline{u_i}v_i$$

You can check for yourself that

$$(\mathbf{u}, \mathbf{v}) = (\overline{\mathbf{v}}, \mathbf{u})$$

and, in particular,

$$(\mathbf{u}, \mathbf{u}) = \sum_{i=1}^{N} \overline{u_i}u_i = \sum_{i=1}^{N} |u_i|^2.$$

Two $N$-dimensional vectors $(\mathbf{u}, \mathbf{v})$ are orthogonal if $(\mathbf{u}, \mathbf{v}) = 0$. Two $N$-dimensional vectors $(\mathbf{u}, \mathbf{v})$ are orthonormal if they are orthogonal and each is of unit length, namely $(\mathbf{u}, \mathbf{u}) = 1$ and $(\mathbf{v}, \mathbf{v}) = 1$. 

Discrete Fourier Transform

We are going to write a function $I(x)$ as a sum of shifted cosine functions

$$I(x) = \sum_{k=0}^{N-1} A(k) \cos\left(\frac{2\pi}{N} k x + \phi(k)\right). \quad (1)$$

To compute the $A(k)$ and $\phi(k)$, we define the $N \times N$ Fourier transform matrix $F$ whose $k^{th}$ row and $x^{th}$ column is:

$$F_{k,x} = \cos\left(\frac{2\pi}{N} k x\right) - i \sin\left(\frac{2\pi}{N} k x\right) \equiv e^{-i \frac{2\pi}{N} k x}$$

The rows of the matrix $F$ have a real part and an imaginary part. The real part is a sampled cosine function. The imaginary part is a sampled sine function. Note that the leftmost and rightmost column of the matrix ($x = 0$ and $x = N - 1$) are not identical. You would need to go to $x = N$ to reach the same value as at $x = 0$. But $x = N$ is not represented.

Multiplying $I(x)$ by the Fourier transform matrix $F$ defines:

$$\hat{I}(k) \equiv F \cdot I(x). \quad (2)$$

$\hat{I}(k)$ is the discrete Fourier transform (DFT) of $I(x)$. Later I will show that

$$\hat{I}(k) = A(k) e^{i\phi(k)}$$

where $A(k)$ and $\phi(k)$ are as in Eq. (1) above.

Example 1: Impulse function

Recall

$$\delta(x) \equiv \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

Its Fourier transform is

$$\hat{\delta}(k) = \sum_{x=0}^{N-1} \delta(x) e^{i \frac{2\pi}{N} k x}$$

$$= 1 \cdot e^{i \frac{2\pi}{N} k 0}$$

$$= 1$$
This is rather surprising. It says that we can obtain an impulse function by summing a set of cosine functions over all frequencies \( k \in 0, 1, \ldots, N - 1 \). (Note that the phase is 0, i.e. \( \phi(k) = 0 \) for all \( k \), and so there are no sine terms.) Basically, what happens is that at \( x = 0 \) all the cosine functions have the value 1, whereas at other values of \( x \) there are a range of values, some positive and some negative, and these other values cancel each other out.

To try to illustrate what is going on here, I have written a Matlab script

http://www.cim.mcgill.ca/~langer/646/MATLAB/sumOfCosines.m

which shows what happens when you add up all the cosines (top) and sines (bottom) of frequency \( k = 0, \ldots, N - 1 \) for some chosen \( N \). The final plot is shown here

http://www.cim.mcgill.ca/~langer/646/MATLAB/sumOfCosines.jpg

The example is rather subtle. The \( N \) cosine functions that are summed to give the top black continuous curve are \( \cos(\frac{2\pi}{N}kx) \) for \( k = 1, \ldots, N - 1 \). The Matlab variable \( x \) indicated on the axis goes from \( 0, 1, \ldots, NM \). The extra factor \( M \) is used to “oversample” the underlying cosine function so that we can plot a continuous curve. The * points are the \( N \) samples \( x = 0, M, 2M, \ldots, (N-1)M \).

Note that the bottom plot shows the sums of the sine functions. The \( N \) sines functions cancel out at all the * sample points.

**Example 2a: the “complex exponential”** \( h(x) = e^{i\frac{2\pi}{N}k_0x} \)

Eventually below we will compute the Fourier transform of \( \cos(\frac{2\pi}{N}k_0x) \) for some fixed integer \( k_0 \). To do so, we will use the following result:

\[
\mathcal{F} \left( e^{i\frac{2\pi}{N}k_0x} \right) = \delta(k - k_0).
\]

That is,

\[
\sum_{x=0}^{N-1} e^{i \frac{2\pi}{N}k_0x} e^{-i \frac{2\pi}{N}kx} = \begin{cases} N, & k = k_0 \\ 0, & k \neq k_0 \end{cases}
\]

How to derive this? The case \( k = k_0 \) should be obvious since the exponent is just 0 and \( e^0 = 1 \) which we sum \( N \) times.

For the case \( k \neq k_0 \), we can use the following identity which you have all seen before. If \( \gamma \) be any number (real or complex) then

\[
(1 - \gamma) \sum_{m=0}^{N-1} \gamma^m = 1 - \gamma^N.
\]

Applying this identity gives

\[
\sum_{u=0}^{N-1} e^{i \frac{2\pi}{N}(k-k_0)x} = \frac{1 - e^{i \frac{2\pi}{N}(k-k_0)}}{1 - e^{i \frac{2\pi}{N}(k-k_0)}}.
\]

The numerator on the right hand side vanishes because \( k - k_0 \) is an integer and so \( e^{i2\pi(k-k_0)} = 1 \).
What about the denominator? Since \( k \) and \( k_0 \) are both in \( 0, \ldots, N - 1 \) and since we are considering the case that \( k \neq k_0 \), we know that \( |k - k_0| < N \) and so \( e^{-i \frac{2\pi}{N}(k-k_0)} \neq 1 \). Hence the denominator does not vanish. Since the numerator vanishes but the denominator does, we can conclude

\[
\frac{1 - e^{-i \frac{2\pi}{N}(k-k_0)x}}{1 - e^{-i \frac{2\pi}{N}(k-k_0)}} = 0.
\]

This completes the derivation.

**Example 2b: Fourier transform of a constant function**

In the last example, call it \( h(x) = e^{i \frac{2\pi}{N} k_0 x} \), if we take \( k_0 = 0 \) then we just have a constant function, namely

\( h(x) = 1 \).

In this case,

\[
\hat{h}(k) = N\delta(k).
\]

Thus, the Fourier transform of the constant function \( h(x) = 1 \) is a delta function in the frequency domain, namely it has value \( N \) at \( k = 0 \) and has value \( 0 \) for all values of \( k \) in \( 1, \ldots, N - 1 \).

**Examples 3 and 4: cosine and sine**

We use Euler’s equation to write cosine and sine in terms of complex exponentials.

\[
\mathcal{F}\cos\left(\frac{2\pi}{N} k_0 x\right) = \sum_{x=0}^{N-1} \cos\left(\frac{2\pi}{N} k_0 x\right)e^{-i \left(\frac{2\pi}{N} k x\right)}
\]

\[
= \sum_{x=0}^{N-1} \frac{1}{2} (e^{i \frac{2\pi}{N} k_0 x} + e^{-i \frac{2\pi}{N} k_0 x})e^{-i \frac{2\pi}{N} k x}
\]

\[
= \frac{N}{2} (\delta(k_0 - k) + \delta(k_0 + k))
\]

We carry out a similar calculation for \( \mathcal{F}\sin\left(\frac{2\pi}{N} k_0 x\right) \).

\[
\mathcal{F}\sin\left(\frac{2\pi}{N} k_0 x\right) = \sum_{x=0}^{N-1} \sin\left(\frac{2\pi}{N} k_0 x\right)e^{-i \left(\frac{2\pi}{N} k x\right)}
\]

\[
= \sum_{x=0}^{N-1} \frac{1}{2i} (e^{i \frac{2\pi}{N} k_0 x} - e^{-i \frac{2\pi}{N} k_0 x})e^{-i \frac{2\pi}{N} k x}
\]

\[
= \frac{-Ni}{2} (\delta(k_0 - k) - \delta(k_0 + k))
\]

**Conjugacy property of the Fourier transform**

**Claim:** Assuming \( I(x) \) is real, which it is for images,

\[
\overline{I(k)} = \hat{I}(N - k).
\]
Note that this property does not apply to Example 2 since $I(x)$ is not real in that case.

Proof:

\[
\hat{I}(N - k) = \sum_{x=0}^{N-1} I(x)e^{-i\frac{2\pi}{N} (N-k)x} = \sum_{x=0}^{N-1} I(x)e^{i\frac{2\pi}{N} kx}e^{-i2\pi x} = \sum_{x=0}^{N-1} I(x)e^{i\frac{2\pi}{N} kx}, \text{ since } e^{2\pi x} = 1 \text{ for any integer } x
\]

\[
= \hat{I}(k)
\]

The conjugacy property clarifies one puzzling aspect of $\hat{I}(k)$ which is that it contains $2N$ values ($N$ complex values), whereas $I(x)$ contains only $N$ values. $\hat{I}(k)$ in fact has only $N$ independent values (all real): once one knows $\hat{I}(k)$ for some $k$, one immediately knows $\hat{I}(N - k)$.

**Inverse Fourier transform**

The inverse of the Fourier transform is defined by a matrix consisting of the conjugate of elements of $F$, divided by the constant $N$:

\[
F^{-1} = \frac{1}{N} \overline{F}
\]

that is, I claim that

\[
\frac{1}{N} \overline{F} F
\]

is the identity matrix.

To prove that $F^{-1}$ as defined above is indeed the inverse of $F$, we index the rows and columns of $\overline{F}$ by $(k_1, x)$ respectively and we index the rows and columns of $F$ by $(x, k_2)$ respectively. We need to show that for any row $k_1$ of $\overline{F}$ and any column $k_2$ of $F$,

\[
\sum_{u=0}^{N-1} e^{i\frac{2\pi}{N} k_1 u} e^{-i\frac{2\pi}{N} k_2 u} = \begin{cases} N, & k_1 = k_2 \\ 0, & k_1 \neq k_2 \end{cases}
\]

But this follows immediately from Example 2 above, since we are doing the same calculation with $k_1$ and $k_2$ here as we did with $k$ and $k_0$ in Example 2.