1. (3 points)

(a) Let \( s(x) = \delta(x - 2) \) be defined on \( x \in \{0, 1, \ldots, N - 1\} \). What is the Fourier transform \( \hat{s}(k) \) of \( s(x) \)?

(b) What is the amplitude spectrum \( |\hat{s}(k)| \) and phase spectrum \( \phi(k) \) of \( s(x) \), where

\[
\hat{s}(k) = |\hat{s}(k)| e^{i\phi(k)}.
\]

**Solution**

(a)

\[
\hat{s}(k) = e^{-i \frac{4\pi}{N} k}
\]

(b)

\[
|\hat{s}(k)| = 1
\]

\[
|\phi(k)| = -\frac{4\pi}{N} k
\]
2. (4 points)
Suppose you have a 2D image that consists of a single Gaussian

\[ I(x, y) = G(x, y, \sigma_1) \]

and you then blur the image with a Gaussian \( G(x, y, \sigma_2) \).

(a) Use the convolution theorem to derive a formula for the resulting image.
(b) Suppose you instead convolve \( I(x, y) \) with a difference of Gaussian function

\[ \text{DOG}(x, y, \sigma_2, \sigma_3) = G(x, y, \sigma_2) - G(x, y, \sigma_3). \]

What is the Fourier transform of the resulting image?

Solution

(a) (2 points)

\[
\mathbf{F} G(x, y, \sigma_1) \ast G(x, y, \sigma_2) = \mathbf{F} G(x, y, \sigma_1) \mathbf{F} G(x, y, \sigma_2) \\
= e^{-\frac{(k_x^2 + k_y^2)\sigma_1^2}{2}} e^{-\frac{(k_x^2 + k_y^2)\sigma_2^2}{2}} \\
= e^{-\frac{(k_x^2 + k_y^2)(\sigma_1^2 + \sigma_2^2)}{2}}
\]

which is the Fourier transform of a Gaussian that has variance \( \sigma^2 = \sigma_1^2 + \sigma_2^2 \). Thus,

\[ G(x, y, \sigma_1) \ast G(x, y, \sigma_2) = G(x, y, \sqrt{\sigma_1^2 + \sigma_2^2}). \]

If you just computed the Fourier transform then you just got 1 point.

(b) (2 points)

\[
\mathbf{F} G(x, y, \sigma_1) \ast (G(x, y, \sigma_2) - G(x, y, \sigma_3)) \\
= \mathbf{F} G(x, y, \sigma_1) \ast G(x, y, \sigma_2) - \mathbf{F} G(x, y, \sigma_1) \ast G(x, y, \sigma_3) \\
= e^{-\frac{(k_x^2 + k_y^2)(\sigma_1^2 + \sigma_2^2)}{2}} - e^{-\frac{(k_x^2 + k_y^2)(\sigma_1^2 + \sigma_3^2)}{2}}
\]
3. (5 points)

Let $Z(x, y)$ be a depth map. Recall the linear shading model,

$$I(x, y) = L_x \frac{\partial Z}{\partial x} + L_y \frac{\partial Z}{\partial y} - L_z .$$

For all questions below, assume the light is from above, so $L_x = 0, L_y > 0, L_z < 0$.

(a) Sketch the image $I(x, y)$ of a Gaussian bump surface of depth $Z(x, y) = G(x, y, \sigma)$.

(b) Now suppose the depth $Z(x, y)$ were a 2D sinusoid. Give an example of the 2D frequency $(k_0, k_1)$ of the depth sinusoid such that the image intensity is constant over $(x, y)$.

(c) Choose a depth sinusoid that is different from the one in (b), so that the intensity now is not constant over $(x, y)$. Suppose this depth sinusoid is translating over time. For which translation vector(s) $(v_x, v_y)$ would there be no change in the image intensity over time? Give a expression for such $(v_x, v_y)$. Also, sketch the image, and draw a vector (or vectors) indicating the direction of motion.

Hint: Recall the motion constraint equation

$$v_x \frac{\partial I}{\partial x} + v_y \frac{\partial I}{\partial y} + \frac{\partial I}{\partial t} = 0 .$$

Solution

See Exercises 3 Question 1.

(a) (2 points)

In the sketch below, the 0 indicate the mean value $-L_z$, the + indicate a value greater than $-L_z$, and - indicates a value less than $-L_z$.

```
0000000000
000+++0000
000+++++000
0000000000
000----0000
0000--00000
0000000000
```

Note $Z(x, y)$ is a depth map, not height map. Bigger values of $Z$ mean the point is further away. I gave only one point if the sketch had the brighter part being on the bottom. Light is from above, so upper part is brighter.

(b) (1 point)

$$Z(x, y) = \sin(k_0 x)$$

for any $k_0$, because the derivative with respect to $y$ is 0.

(c) (2 points)

$$Z(x, y) = \sin(k_1 y)$$

which is a horizontally oriented sinusoid. There is no change in intensity over time if the image velocity is in the $x$ direction, i.e. $(v_x, v_y) = (c, 0)$ for any $c$. 

4. (4 points)

As mentioned in the Held et al paper of A2, “the visual cortex has many neurons with small receptive fields devoted to encoding small disparities and fewer neurons with large receptive fields for encoding large disparities.” In addition, there are many cortical neurons with small receptive fields devoted to encoding spatial structure near the fovea (the center of the image), and fewer neurons with large receptive fields for encoding intensity structure in the periphery.

Using circles to illustrate the size and density of receptive fields, make a sketch of disparity space \((x_l, x_r)\) that illustrates the above statements about how cell receptive field size (circle size) and number (circle density) vary with disparity and eccentricity. Be sure to label the fovea in your sketch.

Solution

The correct solution has size and density decreasing as you move away from the \(d = 0\) line and as you move away from the origin. If you drew the figure with the size decreasing from the \(d = 0\) line, but not decreasing away from the fovea parallel to the \(d = constant\) line, then you got only 3 points.
5. (4 points)
Suppose you are a passenger in a car and you wish to read a detailed road sign on the right side of the road as your car approaches the sign and moves past it. Describe how the three types of eye movements (saccades, smooth pursuit, VOR) are used for you to perform this reading task. In particular, what will be the directions of rotation of the eye relative to the head for the three types of eye movements?
(Note: I am not asking about the net rotation of the eye relative to the head. The net rotation would be the sum of the three rotations above, but would also depend on the head rotation.)

Solution

I generally gave the 4 points as follows:

- The sign is on the right side of the road and so the head will turn to the right as the car drives past.
- Saccades will be used to move the eye toward the sign and then jump from word to word as you read the sign.
- Pursuit is needed to track the sign because the sign is moving relative to the car. Assuming the head is fixed in place for a short time, the eyes will rotate to the right as the sign translates.
- When the head turns to the right to keep the sign close to the center of gaze, the eyes will turn left to cancel the head motion. (The saccades and pursuit will be in addition to this VOR eye rotation.)