lecture 9

Convolution Theorem

Filtering

Wed. Feb. 6, 2013

lecture 8 - Fourier transform

\[ \hat{I}(k) = \sum_{x=0}^{N-1} I(x) e^{-\frac{2\pi i}{N} k x} \]

\[ \hat{I}(k) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{I}(k) e^{\frac{2\pi i}{N} k x} \]

Fourier transform

\[ \hat{I}(k) \equiv \left\{ \begin{array}{l} \cos \left( \frac{2\pi}{N} k x \right) \\ \text{etc.} \end{array} \right\} - i \left\{ \begin{array}{l} \sin \left( \frac{2\pi}{N} k x \right) \\ \text{etc.} \end{array} \right\} \]

\[ \hat{I}(k) \equiv \sum_{x=0}^{N-1} e^{-\frac{2\pi i}{N} k x} I(x) \]

Inverse Fourier transform

\[ I(x) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{I}(k) e^{\frac{2\pi i}{N} k x} \]

\[ I(x) = \frac{1}{N} \mathcal{F} \hat{I}(k) = \mathcal{F}^{-1} \hat{I}(k) \]

Convolution Theorem

Let \( I(x) \) and \( h(x) \) be defined on \( x = 0, 1, \ldots, N-1 \). Then,

\[ F \{ I(x) \ast h(x) \} = F I(x) \cdot F h(x) \]

\[ = \hat{I}(k) \cdot \hat{h}(k) \]

Intuition

\[ I(x) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{I}(k) e^{\frac{2\pi i}{N} k x} \]

Convolve with \( h(x) \)

\[ \text{multiply } \hat{I}(k) \text{ by } \hat{h}(k) \]

\[ I(x) \ast h(x) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{I}(k) \hat{h}(k) e^{\frac{2\pi i}{N} k x} \]
Technical Detail

What if \( h(x) \) is defined on \( x < 0 \) or \( x \geq N \)?

\[
\hat{h}(k) = \sum_{x=0}^{N-1} h(x) e^{-\frac{2\pi i k x}{N}}
\]

Examples:

\[
D(x) = \begin{cases} \frac{1}{2}, & x = -1 \\ -\frac{1}{4}, & x = 1 \\ 0, & \text{otherwise} \end{cases}
\]

\[
B(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ \frac{1}{4}, & x = 1, N-1 \\ 0, & \text{otherwise} \end{cases}
\]

\[
\hat{D}(k) = \sum_{x=0}^{N-1} D(x) e^{-\frac{2\pi i k x}{N}}
\]

\[
= \frac{1}{2} \left( -e^{-\frac{2\pi i k}{N}} + e^{\frac{2\pi i k}{N}} \right)
\]

Euler's equation:

\[
= \frac{1}{2} \left( -e^{-\frac{2\pi i k}{N}} + e^{\frac{2\pi i k}{N}} \right)
\]

\[
= i \sin \left( \frac{2\pi k}{N} \right)
\]

Circular Convolution

for functions defined on integers mod \( N \)

\[
f(x) * I(x) \equiv \sum_{x'=0}^{N-1} f(x') I( (x-x') \mod N )
\]

\[
= \sum_{x'=0}^{N-1} I(x) f( (x-x') \mod N )
\]

i.e. Commutative property still holds.

\[
D_I(x) = \frac{1}{2} (I(x+1) - I(x))
\]

\[
I(x)
\]

\[
D(x) \ast I(x)
\]

boundary effects
\[ B(I(x)) = \frac{1}{4} I(x-1) + \frac{1}{2} I(x) + \frac{1}{4} I(x+1) \]

**Convolution Theorem**

Let \( I(x) \) and \( h(x) \) be defined on \( x = 0, 1, \ldots, N-1 \). Then,

\[ F\{I(x) * h(x)\} = F\{I(x)\} F\{h(x)\} \]

Circular convolution

**Example 1**

Recall \( I(x) * \delta(x) = I(x) \)

\[
\begin{align*}
F(I(x) * \delta(x)) &= F(I(x)) \cdot F(\delta(x)) \\
&= \hat{I}(k) \cdot 1
\end{align*}
\]

**Example 2**

\( B(x) * B(x) * \ldots * B(x) \)

\[ F(B(x) * B(x) * \ldots * B(x)) = \hat{B}(k) \cdot \hat{B}(k) \cdot \ldots \cdot \hat{B}(k) \]

\[ \hat{B}(k) = \frac{1}{2} \left( 1 + \cos\left(\frac{2\pi k}{N}\right) \right) \]
Recall lecture 8: Conjugacy Property

for any $k$, $\hat{I}(k) = \hat{I}(N-k)$

Example

$\hat{B}(k) = \frac{1}{2} (1 + \cos(\frac{2\pi}{N} k))$

Periodicity Property

$\hat{I}(k) = \hat{I}(k + mN)$

for any integer $m$

Why?

$\frac{-i}{N} (k + mN) \times$

$\frac{-i}{N} \frac{2\pi}{N} k x \times$

$\frac{-i}{N} \frac{2\pi}{N} mN x$

$e^{x} = e^{-i \frac{2\pi}{N} k x} = 1$

when $x$ is an integer

Nyquist frequency ($k = N/2$)

Important concept in signal processing but we don’t have time for details.

Basic idea: if the underlying (continuous) signal has frequency higher than $k = N/2$ then these are “aliased” to low frequency components, which produces artifacts.

Example

"Nyquist frequency"

Periodicity Property

$\Rightarrow$ It is common to plot $\hat{f}(k)$ on $k = 1 - \frac{N}{2}, \ldots, -1, 0, 1, \ldots, \frac{N}{2}$

rather than on $k = 0, 1, \ldots, N-1$

Example

$\hat{B}(k)$

$\hat{B}(k) = \frac{1}{2} (1 + \cos(\frac{2\pi}{N} k))$

$\Im(\hat{B}(k))$

$\hat{D}(k)$

$\hat{D}(k) = i \sin(\frac{2\pi}{N} k)$

Example 3

$|F \ D(x) \neq D(x)| = \left| \sin(\frac{2\pi}{N} k) \right|^2$

$|\hat{D}(k)|$
Conjugacy property \( \hat{I}(k) = \hat{I}(N-k) \)

Periodicity property \( \hat{I}(-k) = \hat{I}(N-k) \)

\[ \Rightarrow \hat{I}(k) = \hat{I}(-k) \]

\[ \Rightarrow \text{If we are only interested in } |\hat{I}(k)|, \text{ then it is enough to consider } k = 0, 1, \ldots, \frac{N}{2} \]

\[ \mathcal{F}\{I(x) \ast h(x)\} = \hat{I}(k) \cdot \hat{h}(k) \]

Convolving an image \( I(x) \) with a "filter" \( h(x) \) amplifies (increase) or attenuates (decrease) and possibly phase shifts each "frequency component" of the image.

(Ideal) Low Pass Filter

\[ |\hat{h}(k)| = \begin{cases} 0, & \text{for all } k \text{ such that } 0 < k_0 \leq k \leq \frac{N}{2} \end{cases} \]

(Ideal) High Pass Filter

\[ |\hat{h}(k)| = \begin{cases} 0, & \text{for all } k \text{ such that } 0 \leq k < k_1, \frac{N}{2} \end{cases} \]

(Ideal) Band Pass Filter

\[ |\hat{h}(k)| \neq 0, \text{ for all } k \text{ such that } k_1 < k < k_2 \]
Bandwidth of non-ideal bandpass filter

\[
\text{max } f = \max_{k_1, k_2} \frac{N}{2}
\]

\[\text{bandwidth at half height}\]

\[G(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma^2}\]

Image blurring is commonly modelled by \(I(x) \ast G(x, \sigma)\)
(or in 2D by \(I(x,y) \ast G(x,y, \sigma)\)).

What is \(F G(x, \sigma)\)?

**Fact (proof omitted)**

\[
F \left( \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma^2} \right) \approx e^{-\frac{1}{2} \left( \frac{2\pi\sigma}{N} \right)^2 k^2}
\]

Approximation becomes exact as \(\sigma\) and \(N\) grow (with \(\frac{\sigma}{N}\) fixed)

\[G(x, \sigma) \quad \hat{G}(k, \sigma)\]

\[\text{DOG}(x, \sigma_1, \sigma_2) \quad \text{DOG}(k, \sigma_1, \sigma_2)\]

\[\text{band pass}\]