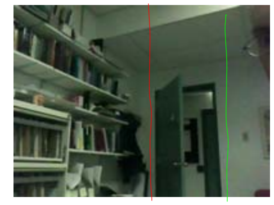
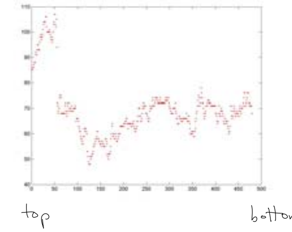
lecture 9

Edge detection

"A Computational Approach to Edge Detection"
John Canny
IEEE Trans. Pattern Analysis and Machine Intelligence
(1986)

Images have noise





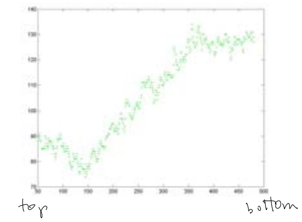


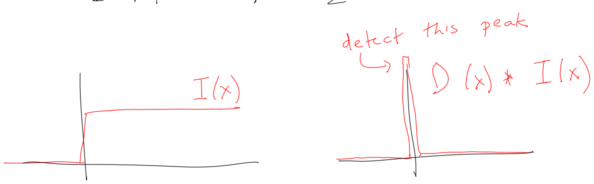
Image irradiance is piecewise smooth on selected columns

But image intensities are not smooth.

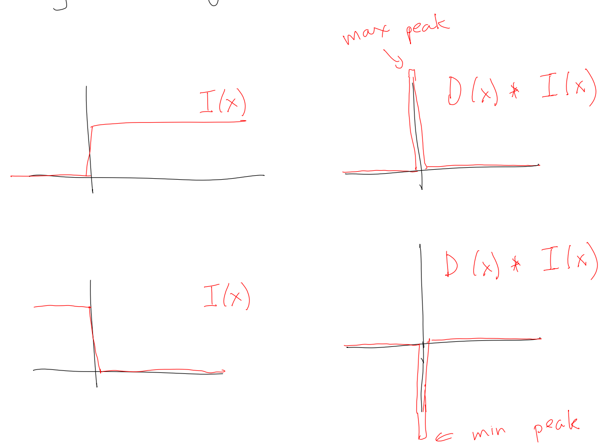
Detecting an edge in a noiseless image

- filter the image $D(x) * I(x)$ and find the maxima

Recall $D(x) * I(x) = \frac{1}{2}\left(I(x+1) - I(x-1)\right)$

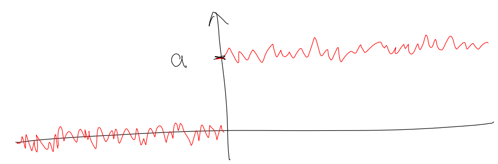


Sign of edge ⟹ max or min



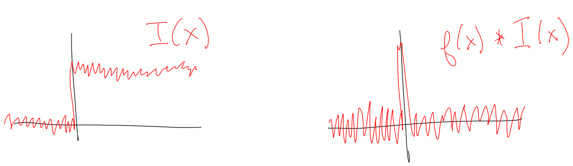
Model of Edge + Noise

$$I(x) = a\,u(x) + n(x)$$



Detecting an edge in a noisy image

- filter the image $f(x) * I(x)$ and find the maxima/minima
- how should you choose $f(x)$ so you can best detect and localize the maxima/minima?



Examples of f(x)

(smoothed version of D(x))

Assumptions about f(x)

- $f(x) = -f(-x)$ anti-symmetric
  in particular $f(0) = 0$.
- $f(x) = 0$ when $|x| > X_{support}$
  i.e. finite support

$$I(x) = a\,u(x) + n(x)$$

"signal" "noise"

$$f(x) * I(x) = a\,f(x) * u(x) + f(x) * n(x)$$

response to signal — response to noise

We want $\dfrac{\text{response to signal}}{\text{response to noise}}$ to be large at the edge.

Response to signal (at $x=0$)

$$a\,u(x) * f(x) = a \int_{-\infty}^{\infty} u(x')\, f(x-x')\, dx'$$

$$= a \int_{0}^{\infty} f(x-x')\, dx'$$

$$a(u * f)(0) = a \int_{-\infty}^{0} f(x')\, dx'$$

Response to noise (at $x=0$)

$$f(x) * n(x) = \sum_{x'=-\infty}^{\infty} f(x')\, n(x-x')$$

$$f * n\,(0) = \sum_{x'=-\infty}^{\infty} f(x')\, n(-x')$$

Assume noise is independent, identically distributed (mean 0, variance σ_n^2)

Statistics

If $X_1, X_2 \ldots X_N$ are independent, identically distributed random variables with mean 0, variance σ_n^2, and a_i for $i = 1, \ldots N$, then:

$E(\;)^2$

$$\text{mean}\left(\sum_{i=1}^{N} a_i X_i\right) = 0$$

$$\text{Var}\left(\sum_{i=1}^{N} a_i X_i\right) = \sigma_n^2 \sum_{i=1}^{N} a_i^2 .$$

See footnote on page 2 of lecture notes.

$$f * n\,(0) = \sum_{x'=-\infty}^{\infty} f(x')\, n(-x')$$

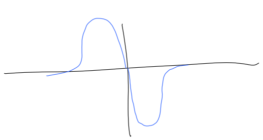
$$\text{Var}\{ f * n(0) \} = \sigma_n^2 \sum_{x'=-\infty}^{\infty} f(x')^2$$

expected value of the square of the response to noise

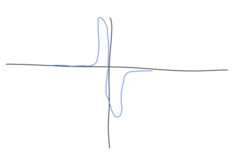
$$\left(\frac{\text{response to signal}}{\text{response to noise}}\right)^2 = \frac{\left(a \sum_{x'=-\infty}^{0} f(x')\right)^2}{\sigma_n^2 \sum_{x'=-\infty}^{\infty} f(x')^2}$$

$$\sim \frac{a^2}{\sigma_n^2}$$

Compare $f(x)$ vs. $f(sx)$



$f(x)$



$f(sx)$

$s > 1$

Assume $f(x)$ is smooth

$$\sum_{x'=-\infty}^{0} f(x') \approx \int_{-\infty}^{0} f(x')\, dx'$$

$$\sum_{x'=-\infty}^{\infty} f(x')^2 \approx \int_{-\infty}^{\infty} f(x')^2\, dx'$$

Compare $f(x)$ vs. $f_s(x) = f(sx)$

$$\int_{-\infty}^{0} f(sx')\, dx' = \frac{1}{s} \int_{-\infty}^{0} f(sx')\, d(sx')$$

$$= \frac{1}{s} \int_{-\infty}^{0} f(w)\, dw, \quad w = sx'$$

$$\int_{x'=-\infty}^{\infty} f(sx')^2\, dx' = \frac{1}{s} \int_{-\infty}^{\infty} f(sx')^2\, d(sx')$$

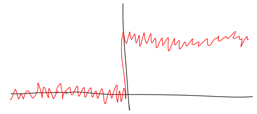
$$= \frac{1}{s} \int_{-\infty}^{\infty} f(w)^2\, dw$$

Compare $f(x)$ vs. $f(sx)$

$$\left(\frac{\text{response to signal}}{\text{response to noise}}\right)^2_{f(sx)} = \frac{\left(a\int_{-\infty}^{0} f(sx')\,dx\right)^2}{\sigma_n^2 \sum_{x'=-\infty}^{\infty} f(sx')^2}$$

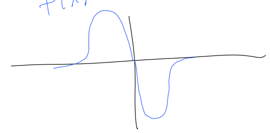
$$\approx \frac{\left(\frac{1}{s}\right)^2 a^2 \int_{-\infty}^{0} f(x')\,dx}{\frac{1}{s}\,\sigma_n^2 \int_{-\infty}^{\infty} f(x')^2\,dx}$$

$$\approx \frac{1}{s}\left(\frac{\text{response to signal}}{\text{response to noise}}\right)^2_{f(x)}$$

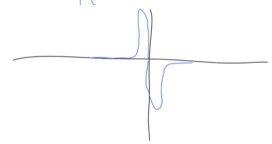


$I(x) = \underset{\text{signal}}{a\,u(x)} + \underset{\text{noise}}{n(x)}$

$f(x)$

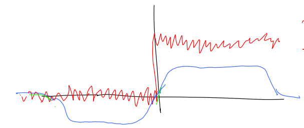


$f(sx)$



$$\frac{1}{s}\left(\frac{\text{response to signal}}{\text{response to noise}}\right)^2_{f(x)} \approx \left(\frac{\text{response to signal}}{\text{response to noise}}\right)^2_{f(sx)}$$

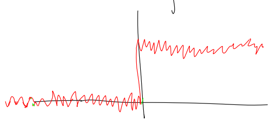
A large filter allows you to average out the (mean zero) noise.



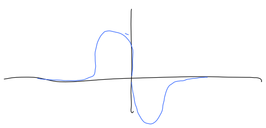
$I(x) = a\,u(x) + n(x)$

$\leftarrow f(x) \rightarrow$

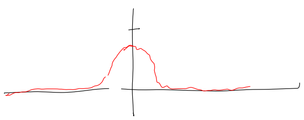
Edge localization



$I(x) = \underset{\text{signal}}{a\,u(x)} + \underset{\text{noise}}{n(x)}$



$f(x)$



$I(x) * f(x)$

Edge localization

Find maximal/minima of $I(x) * f(x)$ near $x = 0$.

$$\frac{d}{dx}\, f(x) * (a\,u(x) + n(x)) = 0$$

$$a\,f(x) * \underbrace{\frac{d}{dx}u(x)}_{\delta(x)} + \frac{df(x)}{dx} * n(x) = 0$$

$$a\,f(x) = -\frac{df(x)}{dx} * n(x)$$

Solution at estimated edge location $x = \hat{x}$

Take Taylor expansion $f(x) \approx 0 + f'(0)\cdot x$

$$\Rightarrow \hat{x} \approx \frac{-\frac{df}{dx} * n(\hat{x})}{a\,f'(0)}$$ for estimated edge location

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Edge localization

$$\hat{x} \approx \frac{-\left(\frac{df}{dx} * n\right)(\hat{x})}{a\,f'(0)}$$

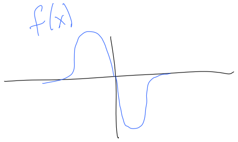
Variance of the numerator will not depend on $\hat{x}$ since noise is independent of the edge location.

$$\therefore \mathrm{Var}(\hat{x}) \approx \frac{\sigma_n^2 \sum_u f'(u)^2}{a^2\, f'(0)^2}$$

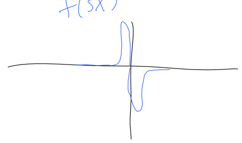
Compare $f(x)$ vs. $f(sx)$, $s > 1$

$$\mathrm{Var}(\hat{x}) \approx \frac{\sigma_n^2 \sum_{u=-\infty}^{\infty} f'(u)^2}{a^2\, f'(0)^2} \approx \frac{\sigma_n^2 \int_{-\infty}^{\infty} f'(u)^2\,du}{a^2\, f'(0)^2}$$

$f(x)$



$f(sx)$

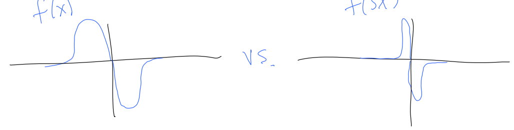


slope of f_s higher but domain smaller (see notes for details)

$$\Rightarrow \mathrm{Var}(\hat{x}) \sim \frac{\sigma_n^2}{a^2}\cdot\frac{1}{s}$$

Edge detection vs. localization

$f(x)$ vs. $f(sx)$



Detection: $\left(\frac{\text{response to signal}}{\text{response to noise}}\right)^2_{f_s} \sim \frac{1}{s}\,\frac{a^2}{\sigma_n^2}$

i.e. better (bigger) when s is small

Localization: $\mathrm{Var}(\hat{x}) \sim \frac{1}{s}\cdot\frac{\sigma_n^2}{a^2}$

i.e. better (smaller) when s is big