lecture 9

Edge detection

"A Computational Approach to Edge Detection"
John Canny
IEEE Trans. Pattern Analysis and Machine Intelligence
(1986)

Images have noise

Image intensity is piecewise smooth on selected columns.
But image intensities are not smooth.

Detecting an edge in a noiseless image

- filter the image $D(x) * I(x)$ and find the maxima.

Recall $D(x) * I(x) = \frac{1}{2} (I(x+1) - I(x-1))$

$D(x) * I(x)$

Sign of edge $\Rightarrow$ max or min

$D(x) * I(x)$

Detect this peak

$I(x)$

More peak

$I(x)$

$I(x)$

$D(x) * I(x)$

$I(x)$

$D(x) * I(x)$

$D(x) * I(x)$

$\text{min peak}$

Example of $f(x)$

(smoothed version of $D(x)$)

Detecting an edge in a noisy image

- filter the image $f(x) * I(x)$ and find the maxima/minima.

- how should you choose $f(x)$ so you can best detect and localize the maxima/minima?

$\text{I(x)}$

$\text{f(x)*I(x)}$

Model of Edge + Noise

$I(x) = a \ u(x) + n(x)$

Assumptions about $f(x)$

- $f(x) = - f(-x)$ anti-symmetric
  in particular $f(0) = 0$.
- $f(x) = 0 \text{ when } |x| > \text{ support}$
  i.e., finite support.
\[ I(x) = a u(x) + n(x) \]

"signal" "noise"

\[ f(x) \ast I(x) = a f(x) \ast u(x) + f(x) \ast n(x) \]

response to signal response to noise

We want response to signal to be large, response to noise to be small, at the edge.

Response to signal (at \( x = 0 \))

\[ a u(x) \ast f(x) = \int_{-\infty}^{\infty} f(x - x') dx' \]

\[ = a \int_{-\infty}^{\infty} f(x-x') dx' \]

\[ a (u \ast f)(0) = a \int_{-\infty}^{\infty} f(x') dx' \]

Response to noise (at \( x = 0 \))

\[ f(x) \ast n(x) = \sum_{x = -\infty}^{\infty} f(x') n(x-x') \]

\[ f \ast n (0) = \sum_{x = -\infty}^{\infty} f(x') n(-x') \]

Assume noise is independent, identically distributed (mean \( 0 \), variance \( \sigma_n^2 \)).

Statistics

If \( X_1, X_2, \ldots, X_N \) are independent, identically distributed random variables, with mean \( \mu \), variance \( \sigma^2 \), and \( a_i \) for \( i = 1, 2, \ldots, N \), then:

\[ \text{mean} \left( \sum_{i=1}^{N} a_i X_i \right) = \mu \]

\[ \text{Var} \left( \sum_{i=1}^{N} a_i X_i \right) = \sigma^2 \sum_{i=1}^{N} a_i^2 \]

See footnote on page 2 of lecture notes.

Compare \( f(x) \) vs. \( f(sx) \)

\[ f(x) \]

\[ f(sx) \]

\[ s > 1 \]

Assume \( f(x) \) is smooth

\[ \int_{-\infty}^{0} f(sx') dx' = \int_{-\infty}^{0} f(x) dx \]

\[ = \int_{-\infty}^{0} f(x) dx' \]

\[ = \int_{-\infty}^{0} f(x')^2 dx' \]

\[ = \int_{-\infty}^{0} f(x)^2 dx \]

Compare \( f(x) \) vs. \( f(sx) = f(sx) \)

\[ \int_{-\infty}^{0} f(sx) dx' = \int_{-\infty}^{0} f(sx') dx' \]

\[ = \int_{-\infty}^{0} f(x) dx \]

\[ = \int_{-\infty}^{0} f(x)^2 dx \]

\[ = \int_{-\infty}^{0} f(x) dx' \]

\[ = \int_{-\infty}^{0} f(x') dx' \]

\[ = \int_{-\infty}^{0} f(x)^2 dx \]
**Compare** $f(x)$ vs. $f(5x)$

$$\frac{\text{response to signal}}{\text{response to noise}} = \frac{\lim_{x \to \infty} f(5x)}{\lim_{x \to \infty} f(x)}$$

$$= \frac{\left(\frac{1}{5}\right) \int_{-\infty}^{\infty} a \delta_{5x} f(x) \, dx}{\int_{-\infty}^{\infty} a \delta_{5x} f(x) \, dx}$$

$$= \frac{1}{5} \frac{\int_{-\infty}^{\infty} a \delta_{5x} f(x) \, dx}{\int_{-\infty}^{\infty} a \delta_{3x} f(x) \, dx}$$

**Edge localization**

$$I(x) = a u(x) + n(x)$$

**Edge localization**

$$\frac{\partial}{\partial x} f(x) \left( a u(x) + n(x) \right) = 0$$

$$a f(x) \frac{\partial}{\partial x} u(x) + \frac{\partial}{\partial x} (n(x)) = 0$$

$$a f(x) \frac{\partial}{\partial x} u(x) + \frac{\partial}{\partial x} n(x) = 0$$

Solution at estimated edge location $x = \hat{x}$

$$\hat{x} = -\frac{\frac{\partial}{\partial x} n(x)}{a f(x)}$$

Variance of the estimator will not depend on $\sigma_n^2$ since noise is independent of edge location.

$$\text{Var} (\hat{x}) = \frac{\sigma_n^2 \int \frac{f(x)^2}{a^2 f'(x)^2}}{a^2 \int f(x)^2}$$

**Edge detection vs. localization**

$$f(x)$$ vs. $f(5x)$

**Edge detection vs. localization**

Detection:

$$\frac{\text{response to signal}}{\text{response to noise}} \approx \frac{1}{5} \frac{a^2}{\sigma_n^2}$$

is better (bigger) when $S$ is small.

Localization:

$$\text{Var} (\hat{x}) \approx \frac{1}{S} \frac{\sigma_n^2}{a^2}$$

i.e. better (smaller) when $S$ is big.