lecture 8

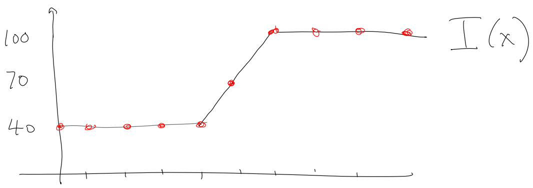
convolution

Motivation - edge detection

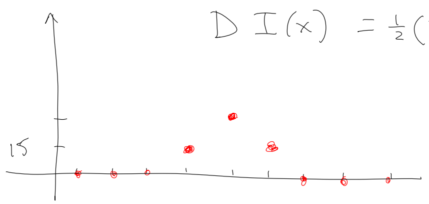
Local difference Operator

$$D\,I(x) \equiv \frac{1}{2}\left(I(x+1) - I(x-1)\right)$$

$$\approx \frac{d}{dx} I(x)$$

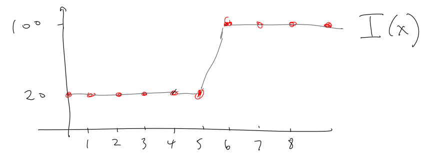


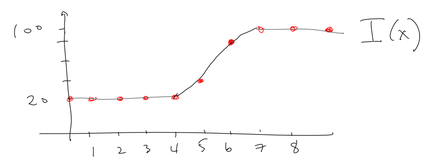
$$D\,I(x) = \frac{1}{2}\left(I(x+1) - I(x-1)\right)$$



Local Average

$$B\,I(x) = \frac{1}{4} I(x-1) + \frac{1}{2} I(x) + \frac{1}{4} I(x+1)$$





Convolution

$$f(x) * I(x) \equiv \sum_{x'} f(x')\, I(x-x')$$

$$= \ldots + \ldots$$
$$+ f(1)\, I(x-1)$$
$$+ f(0)\, I(x)$$
$$+ f(-1)\, I(x+1) + \ldots$$

Convolution

$$f(x) * I(x) \equiv \sum_{x'} f(x')\, I(x-x')$$

Cross Correlation

$$f(x) \circ I(x) \equiv \sum_{x'} f(x')\, I(x+x')$$

$$f(x) * I(x) \equiv \sum_{x'} f(x')\, I(x-x')$$

$$D I = \frac{1}{2}\left(I(x+1) - I(x-1)\right)$$

$$f(x) = \begin{cases} -\frac{1}{2}, & x = 1 \\ \frac{1}{2}, & x = -1 \\ 0, & \text{otherwise} \end{cases}$$

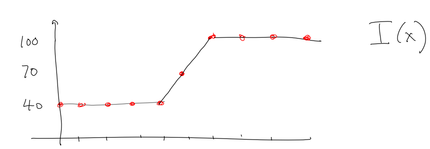
$$B I(x) = \frac{1}{4} I(x-1) + \frac{1}{2} I(x) + \frac{1}{4} I(x+1)$$

$$f(x) = \begin{cases} \frac{1}{4}, & x = 1 \\ \frac{1}{2}, & x = 0 \\ \frac{1}{4}, & x = -1 \\ 0, & \text{otherwise} \end{cases}$$

Boundary Conditions

$$f(x) * I(x) \equiv \sum_{x'=-\infty}^{\infty} f(x')\, I(x-x')$$

If $I(x)$ is defined on $x \in \{0, \ldots N-1\}$ then you can "pad" $I(x)$ with zeros outside of $\{0, \ldots N-1\}$. Other possibilities for padding exist.



Periodic Boundary Conditions

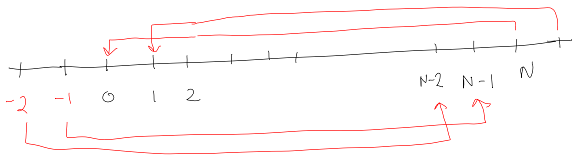
If $I(x)$ is defined on $x \in \{0, \dots N-1\}$.

Pretend that signal $I(x)$ is periodic.

$I(0) \equiv I(-N) \equiv I(N) \equiv I(2N)$ etc.

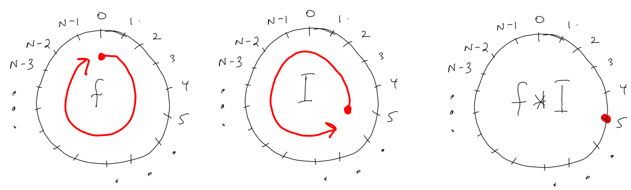
$I(1) \equiv I(1-N) \equiv I(N+1) \equiv$ etc.

$I(2) \equiv I(2-N) \equiv I(N+2) \equiv$ etc.



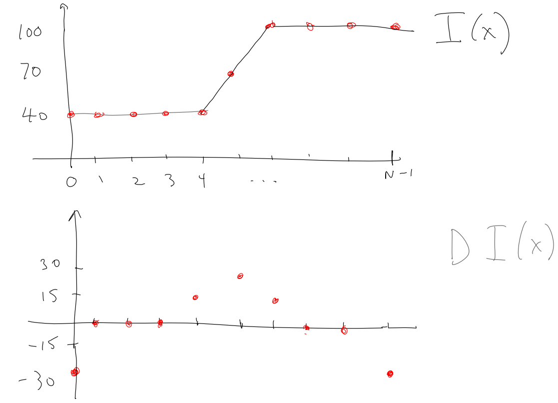
Circular Convolution

$$f(x) * I(x) \equiv \sum_{x'=0}^{N-1} f(x' \bmod N)\, I((x-x') \bmod N)$$



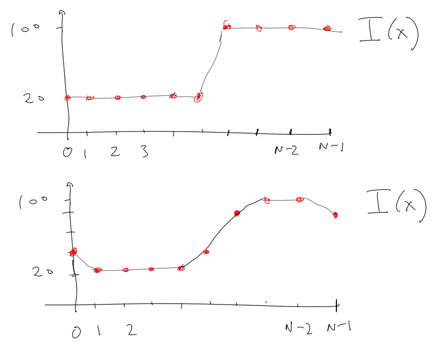
e.g. $x = 5$

$D\, I(x) = \frac{1}{2}\left(I(x+1) - I(x-1)\right)$



Local Average

$$B\, I(x) = \frac{1}{4} I(x-1) + \frac{1}{2} I(x) + \frac{1}{4} I(x+1)$$



Convolution is Commutative

$f(x) * I(x) = I(x) * f(x)$

Proof:

$f(x) * I(x)$

$= \sum_{x'=-\infty}^{\infty} f(x')\, I(x-x')$ Let $x - x' = w$

$= \sum_{w=-\infty}^{\infty} f(x-w)\, I(w)$

$= I(x) * f(x)$

Circular Convolution is Commutative

$f(x) * I(x) = I(x) * f(x)$

Proof:

$f(x) * I(x)$

$= \sum_{x'=0}^{N-1} f(x' \bmod N)\, I((x-x') \bmod N)$ Let $x - x' = w$.

$= \sum_{w=x}^{x-(N-1)} f((x-w) \bmod N)\, I(w \bmod N)$

$= \sum_{w=0}^{N-1} f((x-w) \bmod N)\, I(w \bmod N) = I(x) * f(x)$

Cross correlation is not commutative

Proof: $f(x) \circ I(x) \overset{?}{=} I(x) \circ f(x)$

$f(x) \circ I(x)$

$= \sum_{x'=-\infty}^{\infty} f(x')\, I(x+x')$ Let $x + x' = w$.

$= \sum_{w=-\infty}^{\infty} f(w-x)\, I(w)$

not the same thing

$\neq \sum_{w=-\infty}^{\infty} f(w+x)\, I(w) = I(x) \circ f(x)$

Convolution is Associative

$$(f(x) * g(x)) * h(x) = f(x) * (g(x) * h(x))$$

Example

$$(D(x) * B(x)) * I(x) = D(x) * (B(x) * I(x))$$

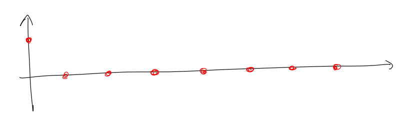
Convolution is Distributive

$$(f(x) + g(x)) * h(x) = f(x) * h(x) + g(x) * h(x)$$

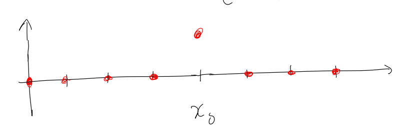
Proof is straight forward

Impulse Function

$$\delta(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$



$$\delta(x - x_0) = \begin{cases} 1, & x = x_0 \\ 0, & \text{otherwise} \end{cases}$$



f(x) is called a "Impulse Response". Why?

$$f(x) * \delta(x) \equiv \sum_{x'=0}^{N-1} f(x')\,\delta((x-x') \bmod N) = f(x)$$

$$f(x) * \delta(x-x_0) = f(x-x_0)$$

Convolution as a sum of shifted functions

$$I(x) = \sum_{x'} I(x')\,\delta(x-x') = I(x) * \delta(x)$$

Convolution of $I(x)$ with $f(x)$ is a sum of shifted impulse responses, i.e. sum of responses to shifted impulses.

$$f(x) * I(x) = f(x) * \sum_{x'} I(x')\,\delta(x-x')$$

$$= \sum_{x'} I(x')\,(f(x) * \delta(x-x'))$$

$$= \sum_{x'} I(x')\, f(x-x') = I(x) * f(x)$$

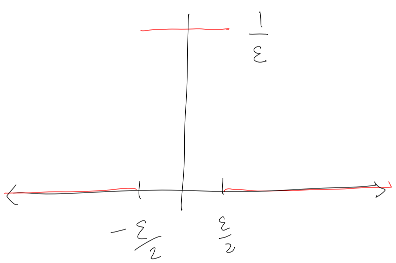
Continuous Convolution

$$I(x) * f(x) = \int_{-\infty}^{\infty} I(x')\, f(x-x')\, dx'$$

Interpretation: add up infinitely many versions of $f(x-x')$, each weighted by $I(x')\,dx$.

Continuous Impulse Function

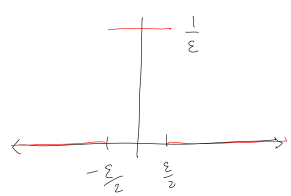
$$\delta_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}, & |x| < \frac{\varepsilon}{2} \\ 0, & \text{otherwise} \end{cases}$$



$$\int \delta_\varepsilon(x)\,dx = 1$$

Impulse Function $\delta(x)$

$$\delta_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}, & |x| < \frac{\varepsilon}{2} \\ 0, & \text{otherwise} \end{cases} \qquad \delta(x) = \lim_{\varepsilon \to 0} \delta_\varepsilon(x)$$



$$\int \delta(x)\,dx = \lim_{\varepsilon \to 0} \int \delta_\varepsilon(x)\,dx = \lim_{\varepsilon \to 0} 1 = 1$$

Continuous Impulse Response Function $f(x)$

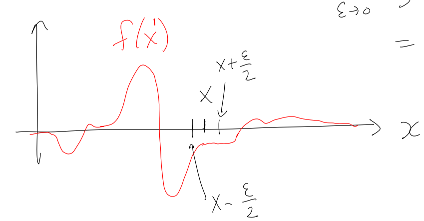
$$f(x) * \delta(x) = f(x)$$

Why?

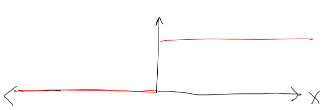
$$f(x) * \delta(x) = \int f(x')\,\delta(x-x')\,dx'$$

$$= \lim_{\varepsilon \to 0} \int f(x')\,\delta_\varepsilon(x-x')\,dx'$$

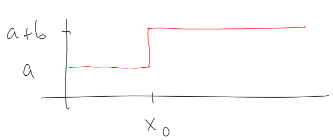
$$= f(x)$$



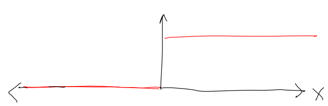
Unit step function $u(x)$



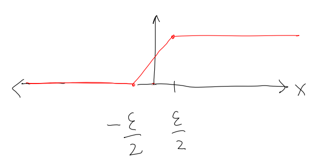
$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



$$I(x) = a + b\,u(x-x_0)$$



$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



$$u_\varepsilon = \begin{cases} 1, & x \geq \frac{\varepsilon}{2} \\ \frac{1}{2} + \frac{1}{\varepsilon}x, & |x| < \frac{\varepsilon}{2} \\ 0, & x \leq -\frac{\varepsilon}{2} \end{cases}$$

$$\frac{d}{dx} u_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}, & |x| < \frac{\varepsilon}{2} \\ 0, & \text{otherwise} \end{cases} = \delta_\varepsilon(x)$$

$$\Longrightarrow \delta(x) = \frac{d}{dx} u(x)$$

Continuous vs. Discrete Derivatives

Let $g(x)$ be differentiable at x.

$$\frac{d}{dx} g(x) = \lim_{\varepsilon \to 0} \frac{g(x+\varepsilon) - g(x-\varepsilon)}{2\varepsilon}$$

$$D\,g(x) = \frac{g(x+1) - g(x-1)}{2}$$