lecture 4

homogeneous coordinates (continued)

+ Camera model

Homogeneous Coordinates: Points at infinity

$$\begin{bmatrix} R\begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ 0 \end{bmatrix} = \left[\begin{array}{c|c} R & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix}$$

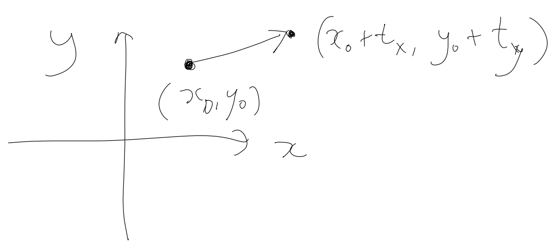
$$\begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix} = \left[\begin{array}{c|c} I & \begin{matrix} t_x \\ t_y \\ t_z \end{matrix} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix}$$

Scaling transformations

$$\begin{bmatrix} \sigma_x X \\ \sigma_y Y \\ \sigma_z Z \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_x & 0 & 0 & 0 \\ 0 & \sigma_y & 0 & 0 \\ 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

2D Homogeneous Coordinates

translate
$$\begin{bmatrix} x_0 + t_x \\ y_0 + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix}$$



2D Homogeneous Coordinates

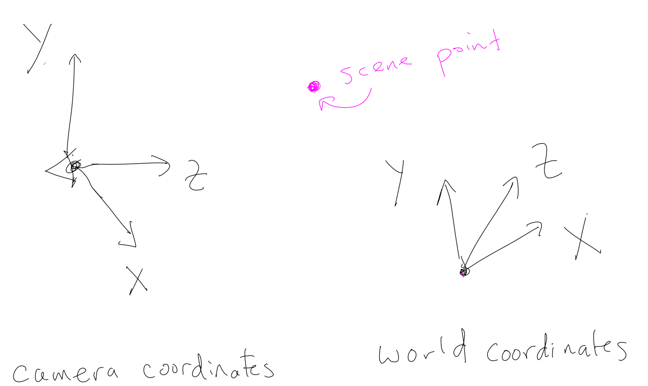
rotate
$$\begin{bmatrix} R\begin{bmatrix} x \\ y \end{bmatrix} \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

scale + shear
$$\begin{bmatrix} \sigma_x x + s \\ \sigma_y y \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_x & s & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Points at infinity

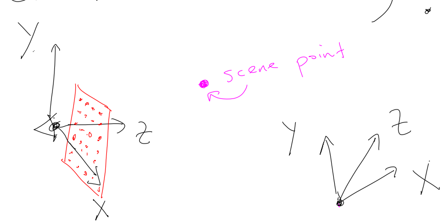
$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \lim_{\varepsilon \to 0} \begin{bmatrix} x \\ y \\ \varepsilon \end{bmatrix} = \lim_{\varepsilon \to 0} \begin{bmatrix} x/\varepsilon \\ y/\varepsilon \\ 1 \end{bmatrix}$$

World vs. Camera Coordinates



Camera Model

How do 3D scene positions (in world coordinates) map to pixel positions (in camera coordinates)?



How do 3D scene positions (in world coordinates) map to pixel positions (in camera coordinates)?

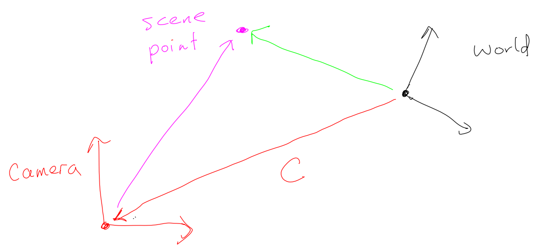
1.) Map from world to camera coordinates

2.) Project onto projection plane

3.) Map from projection plane coordinates to pixel coordinates

1.) Map from world to camera coordinates

Let C be camera position in world coordinates



1.) Map from world to camera coordinates

Let C be camera position in world coordinates

$$\begin{bmatrix}X_c\\Y_c\\Z_c\end{bmatrix} = R_{c\leftarrow w}\left(\begin{bmatrix}X_w\\Y_w\\Z_w\end{bmatrix} - C\right)$$

Q: what are rows/columns of $R_{c\leftarrow w}$?

... in homogeneous coordinates:

$$\begin{bmatrix}X_c\\Y_c\\Z_c\\1\end{bmatrix} = \begin{bmatrix}R_{c\leftarrow w} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix} I & -C\\ 0 & 1\end{bmatrix}\begin{bmatrix}X_w\\Y_w\\Z_w\\1\end{bmatrix}$$

$$= \left[\begin{array}{c|c} R & -RC\\ \hline 0 & 1\end{array}\right]\begin{bmatrix}X_w\\Y_w\\Z_w\\1\end{bmatrix}$$

Typically we simplify

$$\begin{bmatrix}X_c\\Y_c\\Z_c\\ \cancel{1}\end{bmatrix} = \underbrace{\begin{bmatrix}R_{c\leftarrow w} & \cancel{0}\\ \cancel{0} & \cancel{1}\end{bmatrix}}_{4\times4}\underbrace{\begin{bmatrix} I & -C\\ \cancel{0} & \cancel{1}\end{bmatrix}}_{4\times4}\begin{bmatrix}X_w\\Y_w\\Z_w\\1\end{bmatrix}$$

$$\begin{bmatrix}X_c\\Y_c\\Z_c\end{bmatrix} = \underbrace{R_{c\leftarrow w}}_{3\times3}\underbrace{\begin{bmatrix} I & -C\end{bmatrix}}_{3\times4}\begin{bmatrix}X_w\\Y_w\\Z_w\\1\end{bmatrix}$$

2.) Projection onto $Z=f$ plane

Recall $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix} f\frac{X}{Z}\\ f\frac{Y}{Z}\end{bmatrix}$

So $\begin{bmatrix}x\\y\\1\end{bmatrix} = \begin{bmatrix}f\frac{X}{Z}\\ f\frac{Y}{Z}\\ 1\end{bmatrix} \equiv \begin{bmatrix}fX\\ fY\\ Z\end{bmatrix}$

Projection matrix

$$\underbrace{\begin{bmatrix}fX\\fY\\Z\end{bmatrix}}_{\text{2D homogeneous coord}} = \begin{bmatrix} f & 0 & 0 & 0\\ 0 & f & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\underbrace{\begin{bmatrix}X\\Y\\Z\\1\end{bmatrix}}_{\text{3D homogeneous coord}}$$

Projection is not invertible.

$$\begin{bmatrix}fX\\fY\\Z\end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0\\ 0 & f & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}X\\Y\\Z\\1\end{bmatrix}$$

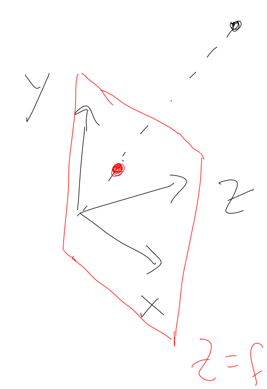
What is the null space?

1.) Map from world to camera coordinates

2.) Project onto projection plane

3.) Map from projection plane coordinates to pixel coordinates

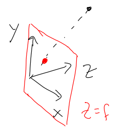
pixels ← proj plane ← Camera coord ← World coord



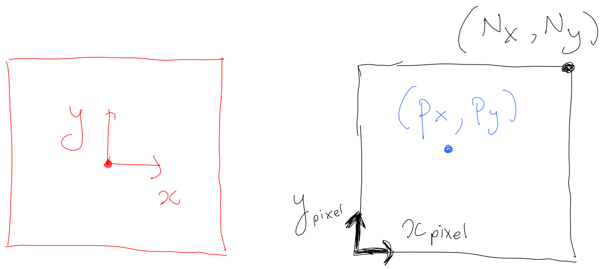
Units for image position

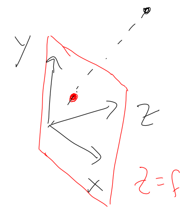
- metres?
- millimetres? (mm)
- pixels?

Let (x, y) have units mm.



How to convert to pixel units?





m pixels per mm

(p_x, p_y) is "principal point"

$$\begin{pmatrix} x_{pixel} \\ y_{pixel} \\ 1 \end{pmatrix} = \begin{pmatrix} m & 0 & p_x \\ 0 & m & p_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

More general .....

$$\begin{pmatrix} w\, x_{pixel} \\ w\, y_{pixel} \\ w \end{pmatrix} = \begin{pmatrix} m_x & 0 & p_x \\ 0 & m_y & p_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$m_x \neq m_y$ means pixel lattice is rectangular, not square

Camera Calibration Matrix $K_{3 \times 4}$

$$\begin{pmatrix} w\, x_{pixel} \\ w\, y_{pixel} \\ w \end{pmatrix} = \overbrace{\begin{pmatrix} m_x & 0 & p_x \\ 0 & m_y & p_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{K} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Camera Calibration Matrix $K_{3 \times 3}$ (more general)

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

s allows for a shear

$\alpha_x = f m_x \; , \; \alpha_y = f m_y$

pixel coord $\leftarrow$ proj plane coord $\leftarrow$ Camera coord $\leftarrow$ world coord

intrinsic extrinsic

$$\underset{3\times1}{\begin{bmatrix} wX \\ wy \\ w \end{bmatrix}} = \underset{3\times3}{K}\; \underset{3\times3}{R}\; \underset{3\times4}{\left[I \,|\, -C \right]} \underset{4\times1}{\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}_w}$$

"Finite Projective Camera" P

(camera is not at infinity)

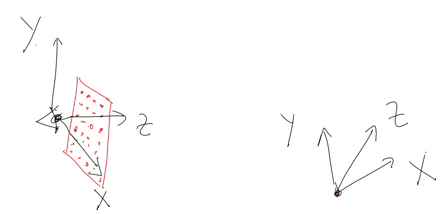
intrinsic extrinsic

$$\begin{bmatrix} wX \\ wy \\ w \end{bmatrix} = \underbrace{\underset{3\times3}{K}\; \underset{3\times3}{R}\; \underset{3\times4}{\left[I \,|\, -C \right]}}_{P} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}_{world\ (w)}$$

Two ways of writing a projection

$$\underset{3\times1}{\begin{bmatrix} wX \\ wy \\ w \end{bmatrix}} = \underset{3\times3}{K}\; \underset{3\times3}{R}\; \underset{3\times4}{\left[I \,|\, -C \right]} \underset{4\times1}{\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}_w}$$

$$\underset{3\times1}{\begin{bmatrix} wX \\ wy \\ w \end{bmatrix}} = \underset{3\times4}{\left[K \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right]} \underset{4\times4}{\left[\begin{array}{c|c} R & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline 000 & 1 \end{array} \right]} \underset{4\times4}{\left[\begin{array}{c|c} I & -C \\ \hline 000 & 1 \end{array} \right]} \underset{4\times1}{\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}_w}$$



camera coordinates world coordinates

$$\underset{3\times4}{P} = \underset{3\times3}{K}\; \underset{3\times3}{R}\; \underset{3\times4}{\left[I \,|\, -C \right]}$$