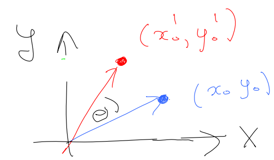
lecture 3

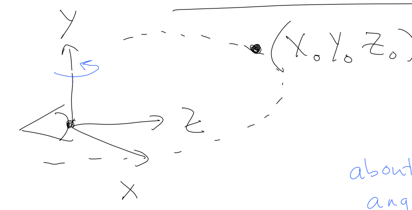
camera rotation

2D Rotation



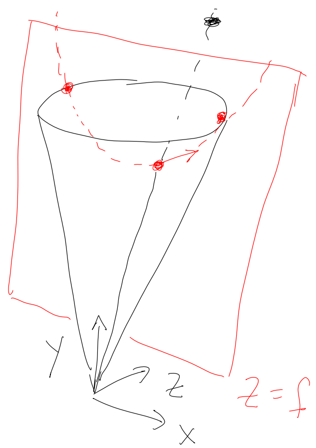
$$\begin{pmatrix} x_0' \\ y_0' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Camera rotation (Y)



about Y axis by angle Ωt

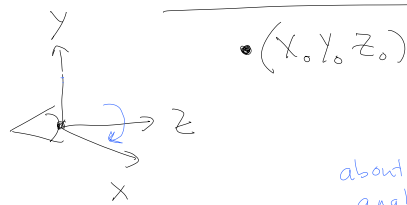
$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} \cos\Omega t & 0 & -\sin\Omega t \\ 0 & 1 & 0 \\ \sin\Omega t & 0 & \cos\Omega t \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}$$



Intersection of a cone $Y = m(x^2 + z^2)$ and a plane $Z = f$ is a parabola.

i.e. $Y = m(x^2 + f^2)$

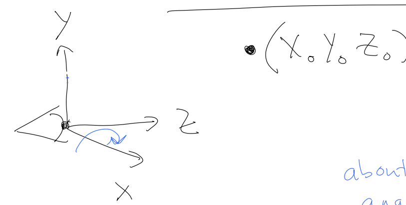
Camera rotation (Z)



about Z axis by angle Ωt

$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}$$

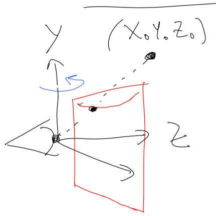
Camera rotation (X)



about X axis by angle Ωt

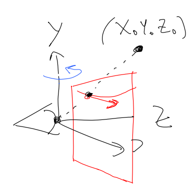
$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega t & -\sin\Omega t \\ 0 & \sin\Omega t & \cos\Omega t \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}$$

Rotation about Y axis



$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} \cos\Omega t & 0 & -\sin\Omega t \\ 0 & 1 & 0 \\ \sin\Omega t & 0 & \cos\Omega t \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} f\,\dfrac{X(t)}{Z(t)} \\[2ex] f\,\dfrac{Y(t)}{Z(t)} \end{pmatrix} = \begin{pmatrix} \dfrac{X_0 \cos\Omega t - Z_0 \sin\Omega t}{X_0 \sin\Omega t + Z_0 \cos\Omega t} \\[2ex] \dfrac{1}{X_0 \sin\Omega t + Z_0 \cos\Omega t} \end{pmatrix} f$$

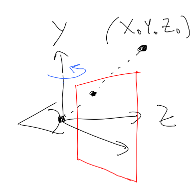


$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} \cos\Omega t & 0 & \sin\Omega t \\ 0 & 1 & 0 \\ -\sin\Omega t & 0 & \cos\Omega t \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}$$

$$V_x = \left.\frac{d\,x(t)}{dt}\right|_{t=0} = \left.\frac{d}{dt}\left(\frac{X_0 \cos\Omega t + Z_0 \sin\Omega t}{-X_0 \sin\Omega t + Z_0 \cos\Omega t}\right)\cdot f\,\right|_{t=0}$$

$$= \frac{\Omega Z_0 Z_0 + X_0 \Omega X_0}{Z_0^2}\cdot f$$

$$= f\Omega\left(1 + \left(\frac{x}{f}\right)^2\right)$$

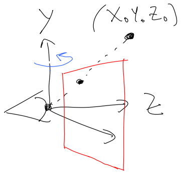


$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} \cos\Omega t & 0 & \sin\Omega t \\ 0 & 1 & 0 \\ -\sin\Omega t & 0 & \cos\Omega t \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}$$

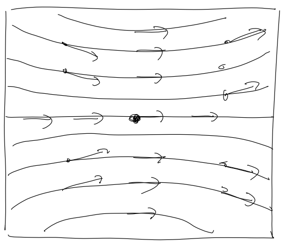
$$V_y = \left.\frac{dy(t)}{dt}\right|_{t=0} = \left.\frac{d}{dt}\left(\frac{Y_0}{-X_0 \sin\Omega t + Z_0 \cos\Omega t}\right)\cdot f\,\right|_{t=0}$$

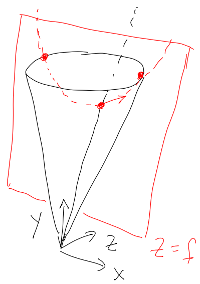
$$= \frac{Y_0 X_0 \Omega}{Z_0^2}\, f$$

$$= \frac{xy}{f}\,\Omega$$

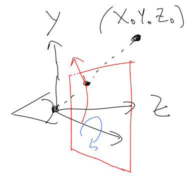


$$(v_x, v_y) = \left(f\left(1+\left(\frac{x}{f}\right)^2\right), \frac{xy}{f}\right)\Omega$$

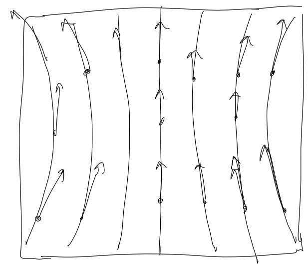




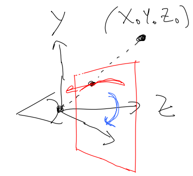
Rotation about X axis ("tilt")



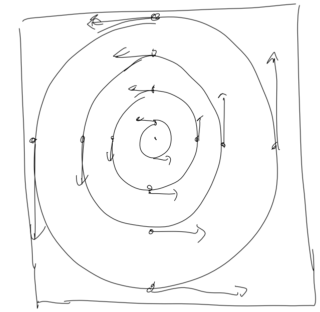
$$(v_x, v_y) = \left(\frac{xy}{f}, f\left(1+\left(\frac{x}{f}\right)^2\right)\right)\Omega$$



Rotation about Z axis ("roll")



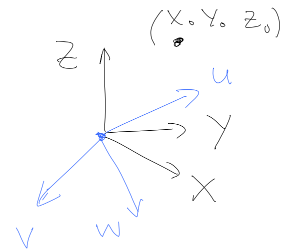
$$(v_x, v_y) = (-y, x)\cdot\Omega$$



- Rotation fields don't depend on Z.
- To define smooth rotations about an arbitrary axis and the resulting image motion field is more complicated. (Details omitted)

Finite Rotations

(brief review of some linear algebra)



Let XYZ and uvw be two orthonormal coordinate systems (with the same origin)

$$\begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} - & u & - \\ - & v & - \\ - & w & - \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}$$

Rotation Matrix R

- $R^T R = I$ so $R^T = R^{-1}$
- $\det R = |R| = 1$
- R preserves angles between vectors

$$(Rv_1)\cdot(Rv_2) = v_1^T R^T R v_2 = v_1^T v_2$$

and thus R preserves lengths of vectors too.

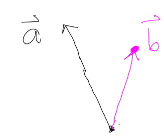
Eigenvectors and eigenvalues of rotation matrix.

$$\lambda v = R v$$

- All eigenvalues have $|\lambda| = 1$
- Eigenvalues are $1, e^{i\theta}, e^{-i\theta}$
- Eigenvector corresponding to $\lambda = 1$ is the axis of rotation

Cross Product (using a matrix)

Let $\vec{a}$ be unit vector



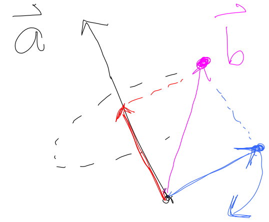
$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y,\ b_x a_z - a_x b_z,\ a_x b_y - a_y b_x)^T$$

$$= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\vec{a}]_\times b$$

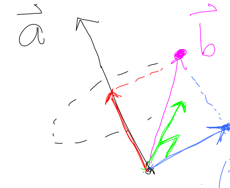
Given unit vector $\vec{a}$ (axis of rotation) and angle θ, how do we construct a rotation about $\vec{a}$ by angle θ?

[Good exercise for understanding rotations.]



$$\vec{b} = \vec{a}(\vec{a}\cdot\vec{b}) + \vec{b} - \vec{a}(\vec{a}\cdot\vec{b})$$

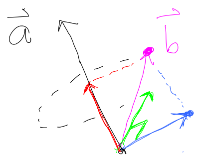
$$= \vec{a}\vec{a}^T\vec{b} + (I - aa^T)\vec{b}$$



$[\vec{a}]_\times (I - \vec{a}\vec{a}^T)\vec{b}$

$(\vec{I} - aa^T)\vec{b}$

These two vectors are orthogonal and span the plane perpendicular to $\vec{a}$.



$[\vec{a}]_\times (I - \vec{a}\vec{a}^T)\vec{b}$

$$\vec{b} = \vec{a}\vec{a}^T b + (I - aa^T)\vec{b}$$

$$R\vec{b} = \vec{a}\vec{a}^T\vec{b} + R(I - aa^T)\vec{b}$$

$$= \text{''} + \cos\theta\,(I - aa^T)\vec{b} + \sin\theta\,[\vec{a}]_\times (I - aa^T)\vec{b}$$

Thus

$$R = \vec{a}\vec{a}^T + \cos\theta\,(I - \vec{a}\vec{a}^T) + \sin\theta\,[\vec{a}]_\times (I - \vec{a}\vec{a}^T)$$

Homogeneous Coordinates

(useful representation for finite translations and finite rotations i.e. not velocities)

translation $\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} \rightarrow \begin{pmatrix} X_0 + t_x \\ Y_0 + t_y \\ Z_0 + t_z \end{pmatrix}$

rotation $\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} \rightarrow \underbrace{\begin{bmatrix} R \end{bmatrix}}_{3\times 3} \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}$

Trick - use 4D instead of 3D

Translation

$$\begin{bmatrix} X_0 + t_x \\ Y_0 + t_y \\ Z_0 + t_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix}$$

Rotation

$$\begin{bmatrix} R\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} R_{3\times 3} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{array}\right] \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix}$$

Define $\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} wX_0 \\ wY_0 \\ wZ_0 \\ w \end{pmatrix}$ where $w > 0$

$\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \\ \varepsilon \end{pmatrix} \equiv \begin{pmatrix} X_0/\varepsilon \\ Y_0/\varepsilon \\ Z_0/\varepsilon \\ 1 \end{pmatrix}$ refers to 3D point $\begin{pmatrix} X_0/\varepsilon \\ Y_0/\varepsilon \\ Z_0/\varepsilon \end{pmatrix}$

$$\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \\ 0 \end{pmatrix} = \lim_{\varepsilon \to 0} \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \\ \varepsilon \end{pmatrix} = \lim_{\varepsilon \to 0} \begin{pmatrix} X_0/\varepsilon \\ Y_0/\varepsilon \\ Z_0/\varepsilon \\ 1 \end{pmatrix}$$

= 3D point at infinity in direction $\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}$