Shading

The term *shading* typically refers to variations in irradiance along a smooth Lambertian surface. Recall that if a surface point is illuminated by parallel light source from direction $\mathbf{l}$, then the surface irradiance at the point is proportional to

$$E(\mathbf{X}) = L_{src} \, \Omega \, \mathbf{n}(\mathbf{X}) \cdot \mathbf{l}.$$

From now on, we will ignore the dependence of $L_{src}\Omega$ since they are constant.

Let's suppose that the surface is Lambertian (recall lecture 6 page 5) and has spatially constant reflectance, that is, the reflectance does not vary with position $\mathbf{x}$ on the surface. An example is a constant color bedsheet or other drapery surface, or clothing with no pattern printed on it. In this case, the above model implies that the radiance of light reflected from the surface varies only with the surface normal. Everything else in the above equation is assumed to be constant.

Weak perspective

To keep the discussion of shading as simple as possible, we consider a relatively small *surface* in the scene. Let XYZ be the usual camera coordinates and assume we can write points on the surface as $(X, Y, \ Z(X, Y))$. We assume that the range of Z values on this surface is small relative to the mean depth Z value, which we call Z_0. That is, we assume $|\frac{Z-Z_0}{Z_0}|$ is near zero for all Z values of points on the surface. For example, suppose the surface is a person's shirt (covering their torso) which is seen from a distance of at least a few meters. The distance between the farthest visible point of the shirt might only be 10 cm greater than the distance to the nearest part of the shirt.

The above assumptions lead to the following *weak perspective* model. For each point (X, Y, Z) on the surface, we have

$$x = f\frac{X}{Z} \approx f\frac{X}{Z_0}$$

$$y = f\frac{Y}{Z} \approx f\frac{Y}{Z_0}.$$

We can thus write surface irradiance $E(X, Y, Z)$ as a function of (X, Y) only (since Z is a function of (X, Y)),

$$E(X, Y) \approx E(\frac{Z_0}{f}x, \frac{Z_0}{f}y) \qquad (1)$$

[ASIDE: This is sometimes called *scaled orthographic projection*, namely we are projecting the points parallel to the Z axis, and then scaling the image plane by a factor $\frac{Z_0}{f}$.]

The reason this last equation is important is that, because the surface is Lambertian, surface irradiance at (X, Y, Z) is proportional to the radiance of light reflected from (X, Y, Z) and so surface irradiance is proportional to image irradiance at (x, y). Thus there is a direct relationship between image irradiance variations and surface irradiance variations. In what follows, we work with surface irradiance only, but keep in mind that this is basically the same as image irradiance for the conditions via Eq. (1).

Surface normal

The surface normal vector is perpendicular to the surface and so is determined by the *depth gradient* $(\frac{\partial Z}{\partial Y}, \frac{\partial Z}{\partial X})$. Specifically, consider a step on the surface from (X, Y, Z) to some other nearby point $(X + \Delta X, Y + \Delta Y, Z + \Delta Z)$ on the surface. Then

$$\Delta Z \approx \frac{\partial Z}{\partial X}\Delta X + \frac{\partial Z}{\partial Y}\Delta Y$$

or

$$(\Delta X, \Delta Y, \Delta Z) \cdot (\frac{\partial Z}{\partial X}, \frac{\partial Z}{\partial Y}, -1) \approx 0 \ .$$

The latter relationship holds for any step $(\Delta X, \Delta Y, \Delta Z)$ along the surface. It follows that the vector $(\frac{\partial Z}{\partial X}, \frac{\partial Z}{\partial Y}, -1)$ is perpendicular to the surface. Hence this vector is in the direction of the *surface normal*. If we rescale this vector to unit length, then we get the *unit normal vector*

$$\mathbf{n} \equiv \frac{1}{\sqrt{(\frac{\partial Z}{\partial X})^2 + (\frac{\partial Z}{\partial Y})^2 + 1}}(\frac{\partial Z}{\partial X}, \frac{\partial Z}{\partial Y}, -1) \ .$$

Note that the Z component of $\mathbf{n}$ is negative, since positive Z goes away from the observer, and so for the surface to be visible the normal must have a negative Z. Here we are talking about the *outward normal*, i.e. out of the object.

The surface irradiance is then

$$E(X, Y) \ = \ L_{src}\Omega\frac{1}{\sqrt{(\frac{\partial Z}{\partial X})^2 + (\frac{\partial Z}{\partial Y})^2 + 1}} \ (\frac{\partial Z}{\partial X}, \frac{\partial Z}{\partial Y}, -1) \cdot (l_X, l_Y, l_Z). \tag{2}$$

where $\mathbf{l} = (l_X, l_Y, l_Z)$ is the unit vector pointing toward the light source.

This model holds only when $\mathbf{n}(X, Y) \cdot \mathbf{l} \ \geq 0$, since it is meaningless to have negative intensities. If the inner product of $\mathbf{n}$ and $\mathbf{l}$ is less than zero, this implies would imply that the surface is facing away from the light source at that point. In this case, the surface would not be illuminated by the source. It would be in shadow, and its illuminance component from the source would be zero (not negative). To keep the notation down, we are not considering this case. But you should understand it is there.

Photometric stereo

See slides for a discussion of this idea.

Bas relief (linear shading)

Let's look at the case of a *bas relief* surface, namely a surface that is nearly flat (planar) except for small hills and valleys. An example is a wrinked shirt or stucco, a coin, the bumps on the bricks on the painted wall in our classroom. We restrict ourselves to the case that the surface slopes $\frac{\partial Z}{\partial X}$ and $\frac{\partial Z}{\partial Y}$ are sufficiently small in magnitude, that one can approximate Eq. (2) by a Taylor series expansion around $(\frac{\partial Z}{\partial X}, \frac{\partial Z}{\partial X}) = (0, 0)$, and we keep terms up to second order.

Recalling from freshman math that $(1+u)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}u +$ higher order terms, we get

$$\frac{1}{\sqrt{1 + (\frac{\partial Z}{\partial X})^2 + (\frac{\partial Z}{\partial Y})^2}} = 1 - \frac{1}{2}\{(\frac{\partial Z}{\partial X})^2 + (\frac{\partial Z}{\partial Y})^2\} + \ H.O.T.$$

We then substitute into Eq. (2):

$$E(X,Y) \quad = \quad \{1 - \frac{1}{2}\{(\frac{\partial Z}{\partial X})^2 + (\frac{\partial Z}{\partial Y})^2\} + \ H.O.T.\} \ (\frac{\partial Z}{\partial X} l_X + \frac{\partial Z}{\partial Y} l_Y - l_Z)$$

If $|\frac{\partial Z}{\partial X}|$ and $|\frac{\partial Z}{\partial Y}|$ are small enough, then we can ignore terms that are higher than first order. This gives:

$$\begin{aligned} E(X,Y) \quad &\approx \quad -l_Z \quad + \quad (\frac{\partial Z}{\partial X} l_X + \frac{\partial Z}{\partial Y} l_Y \ ) \ + \frac{l_Z}{2}((\frac{\partial Z}{\partial X})^2 + (\frac{\partial Z}{\partial Y})^2) \\ &= \quad \text{constant} \quad + \quad \text{linear} \quad + \quad \text{quadratic} \end{aligned}$$

The linear component depends on l_X, l_Y. The quadratic component depends on l_Z. The linear term tends to dominate over the quadratic term if $\mid l_Z \mid \ll |(l_X, l_Y)|$, whereas the quadratic term tends to dominate when $\mid l_Z \mid \gg |(l_X, l_Y)|$. In class, I only mentioned the linear component.

Consider the condition that the light source direction $\mathbf{l}$ is near perpendicular to Z and Y axes of the surface, say $(l_X, l_Y, l_Z) \approx (\sqrt{1-\epsilon^2}, 0, -\epsilon)$. In this case, we have

$$E(X,Y) = -l_Z + \frac{\partial Z}{\partial X} l_X$$

For each Y, we can do a 1D integration to get:

$$\begin{aligned} \int_{X_0} E(X,Y) dX \quad &= \quad -l_Z(X - X_0) + l_X \int_{X_0}^{X} \frac{\partial Z}{\partial X} dX \\ &= \quad -l_Z(X - X_0) + l_X(Z(X) - Z(X_0)) \end{aligned}$$

Thus, given the surface irradiance function, we can say something about the Z values of different points on the surface.

This example should at least give you a sense of how you might solve the general *shape from shading* problem.[1] In the general problem, one tries to solve the "non-linear first order partial differential equation", namely Eq. (1). The solution also involves integration of the partial derivatives of Z along an image curve.

Shape from shading on a cloudy day

Let's now turn to a different shading problem (which I introduced and solved in my PhD thesis). Again suppose we have a Lambertian surface of uniform reflectance. Now, however, rather than having a unique light source direction as on a sunny day, we instead have a diffuse light source such

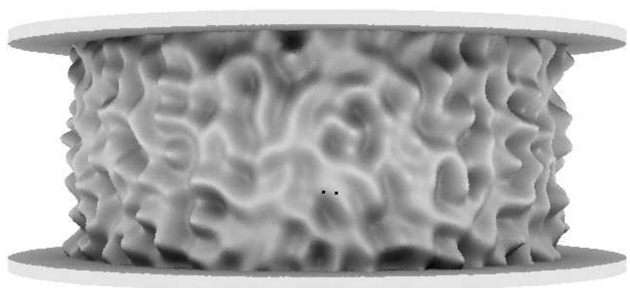
[1] The first person to solve this problem was Berthold Horn at MIT in his Ph.D. thesis in the late 1960's, and there was much followup work on this problem in the 1980s and even some in the 1990s.

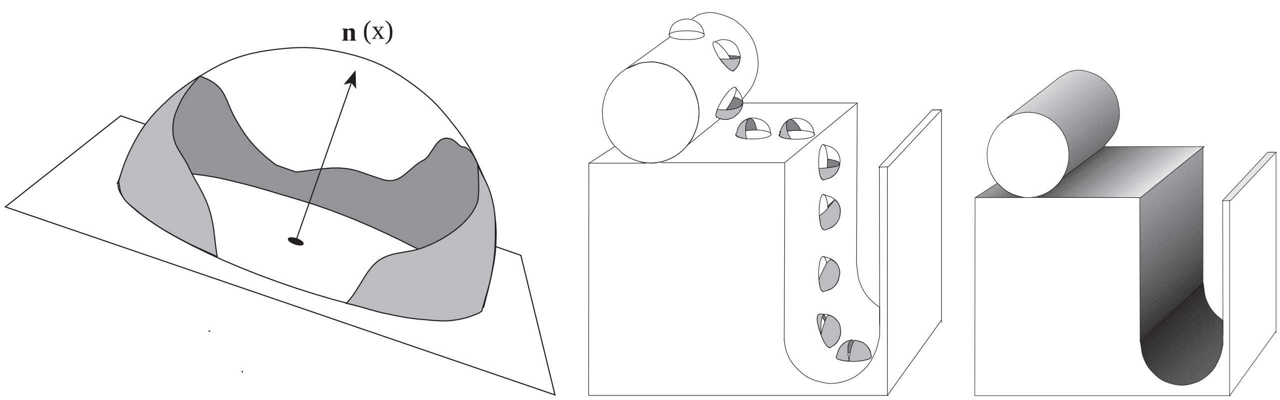
as on a cloudy day. We assume that the radiance from the sky is constant L_{src} and is much greater than the radiance coming from other surfaces in the scene. So we ignore the radiance from surfaces in the scene, and approximate the surface irradiance as

$$E(X,Y) \ = \ L_{src} \int_{\mathbf{l} \in \mathcal{V}(\mathcal{X},\mathcal{Y})} \mathbf{n}(\mathbf{x}) \cdot \mathbf{l} \; d\Omega$$

where $\mathcal{V}(X,Y)$ is the set of directions in which the sky is visible from $(X,Y,Z(X,Y))$. Notice that the irradiance now depends both on the surface normal variations as well as on the amount of sky that is visible.

Below is an image of a bumpy surface rendered with computer graphics, under the diffuse lighting model just given. There are noticable intensity variations on the surface which are due to the hill vs. valley shapes. Below that are three sketches illustrating the causes of shading under diffuse lighting. The hemispheres in the two sketches on the left are partitioned into two regions – a solid color which represents directions in which the sky is *not* visible, and the remainder which are directions in which the sky is visible. Notice that with the example of the cylinder lying on the ground, the points on the of the cylinder sees the entire sky, but as you move toward the contact between the cylinder and the ground, there is less light arriving at the surface and so it is darker.





Having a model of shading is just the first step. We still need an algorithm that estimates the surface shape from such shading. The first algorithm I came up was based on an approximation of the above lighting model, in which the surface irradiance only depends on the solid angle of the sky that is visible. To obtain such an approximation, one can replace the factor $\mathbf{n} \cdot \mathbf{l}$ by $\frac{1}{2}$ which is its average value over the hemisphere[2], giving:

$$E(X,Y) \;\approx\; \frac{L_{src}}{2} \int_{\mathbf{l} \in \mathcal{V}(\mathcal{X},\mathcal{Y})} d\Omega. \tag{3}$$

The integral just gives us the solid angle of the sky that is visible, and hence *we model the surface irradiance as some constant times the solid angle of the visible sky.*

The task now is to come up with an algorithm for computing surface shape, given the surface irradiance function.[3] This shape from shading problem is challenging to solve because the visibility of the sky is not a *local* property of the surface shape (like the surface normal) but rather it is a *global* property. In particular, the sky visibilty $\mathcal{V}(X,Y)$ at a surface point $(X,Y,Z(X,Y))$ can be affected by scene points that are far away, which can "cast shadows" i.e. block the sky.

The algorithm I came up with for solving this shape from shading problem came from thinking of the visibility function not just on the surface but also in the 3D space above the surface. In particular, when the visibility function $\mathcal{V}(X,Y,Z)$ is defined in 3D space, it has strong local constraints, which I called *local visibility constraints*: If the sky is visible from some point (X,Y,Z) in space and in a direction $\mathbf{l}$, then the sky also will be visible from the nearby point $(X,Y,Z)+r\mathbf{l}$ and in direction $\mathbf{l}$.

The algorithm is based on this constraint, and can be explained intuitively as follows. Imagine a surface with height function $Z(X,Y)$. Suppose we were to flood the surface with water so that all points on the surface were covered with water. Then, we drop down on the surface of the water a square grid of water spiders at locations (X,Y), which correspond to the image pixels (under weak perspective). Using the approximate model that irradiance is proportional to the solid angle of the visible sky, we can tell each water spider the following: "when the water is drained away and you land on the surface below, you will see a certain total solid angle of the sky."

The algorithm proceeds depth by depth, analogous to draining away the water. For each depth $Z=k$ and for each point (X,Y,k) at that depth, each water spider that has not yet landed uses the local visibility constraints to compute the set of directions in which the sky is visible. If for any X,Y the solid angle of the sky decreases to the given solid angle (what the water spider is told at the beginning), then the water spider knows that it has reached the ground and it stops. After it stops, it is able to block the sky from other water spiders that have not yet stopped.

How does the water spider compute how much of the sky is visible? The idea is to use the local visibility constraint mentioned earlier. Suppose the water spider at (X,Y,Z) wishes to know if the sky is visible in direction $\mathbf{l}=(l_X,l_Y,l_Z)$. It can then ask the nearest neighboring water spider in direction (l_X,l_Y) whether the sky was visible in direction $\mathbf{l}$ earlier when that water spider passed through a point on the ray $(X,Y,Z)+r\mathbf{l}$. This way, *local visibilities are used to solve a global problem.*

There are various technical details required to get this algorithm to work, outside the (X,Y) range of the image, we assume the surface is flat and so the surface $Z(X,Y)$ has been excavated from the ground – it lies no higher than the (assumed) flat ground outside the field of view.

[2] You can calculate this using spherical coordinates

[3] In fact, we are given the image irradiance, not surface irradiance. But these two quantities are closely related for a Lambertian surface. Details omitted here, since its not the main point.