When defining geometric objects such as points, lines, polygons, etc, we often want to perform transformations such rotation, scaling, and translation. To do so, we perform linear transformations on the points that define the object (i.e. the vertices, in the case of a line or polygon). Let's begin by discussing rotation.

2D Rotations

If we wish to rotate by θ degrees *clockwise*, then we perform a matrix multiplication by

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

namely

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{R} \begin{bmatrix} x \\ y \end{bmatrix}$$

The rotation is clockwise because the vector (1,0) goes to $(\cos\theta, -\sin\theta)$ which has a negative y value if $\theta < 90$ degrees.

There are two ways to think about what's happening. First, the columns of $\mathbf{R}$ are the images of the unit vectors in the x and y directions. Second, multiplying by $\mathbf{R}$ projects the vector (x, y) onto vectors that are defined by the rows of $\mathbf{R}$. The first interpretation says that we are keeping the coordinate system but moving (rotating) the points within this coordinate system. The second says we are defining a new coordinate system to represent our space $\Re^2$.

3D Rotations

Let's consider the same idea for rotations in $\Re^3$. One way to think of a rotation is to move a set of points (x, y, z) in the world relative to some fixed coordinate frame. For example, you might rotate your head or rotate your chair (and your body, if its sitting on the chair). In this interpretation, we define a 3D *axis of rotation* and an *angle of rotation*. We also define the origin $(0, 0, 0)$ of the coordinate system. We rotate *about the origin*, so that the origin doesn't move. For the example of rotating your head, the origin could be some point in the center of your head.

We can rotate about many different axes. For example, to rotate about x axis, we perform:

$$\mathbf{R_x}(\theta) \quad : \quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This transformation leaves the x coordinate of each 3D point fixed, but changes the y and z coordinates. Notice that the xz rotation is essentially a 2D rotation.

Note that its not clear whether this is clockwise or counterclockwise rotation. It depends on how we define these terms in 3D, and whether the xyz coordinate system is right handed or left handed.

The standard coordinate xyz system is *right handed*, in the sense that if we go from thumb x to index finger y to middle finger z, then

$$x \times y = z$$

and

$$x = y \times z.$$

In a right handed coordinate system, the above rotation $\mathbf{R}_x$ is clockwise if we are looking in the $-x$ direction, and is counterclockwise if we are looking in the positive x direction.

We could also rotate about the y axis (leaving y fixed) or about the z axis (leaving z fixed),

$$\mathbf{R_y}(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \mathbf{R_z}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Strangely, the position of the "-" sign is different for the y axis than for the x and z axes. Why? Verify for yourself that if we want the rotation to be clockwise (in the xz plane, when looking in the $-y$ direction) then we need to define the sign as above.

In general, by a *3D rotation matrix*[1], we will just mean a real 3×3 invertible matrix $\mathbf{R}$ such that

$$\mathbf{R}^T = \mathbf{R}^{-1}$$

i.e.

$$\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix, and so the rows (and columns) of $\mathbf{R}$ are orthogonal to each other and of unit length. We also require that $\det \mathbf{R} = 1$ (see below).

Such matrices preserve the length of vectors, as well as the angle between two vectors. To see this, let $\mathbf{p}_1$ and $\mathbf{p}_2$ be two vectors, possibly the same. Then

$$(\mathbf{R}\mathbf{p}_1) \cdot (\mathbf{R}\mathbf{p}_2) = \mathbf{p}_1^T \mathbf{R}^T \mathbf{R} \mathbf{p}_2 = \mathbf{p}_1^T \mathbf{p}_2 = \mathbf{p}_1 \cdot \mathbf{p}_2.$$

If $\mathbf{p}_1 = \mathbf{p}_2$ then we see vector length is preserved. More generally, we see that the cosine of the angle between vectors is preserved, and hence the angle between vectors is preserved.

Why did we require that $\det \mathbf{R} = 1$? An example of an orthogonal matrix that does *not* have this property is:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix performs a mirror reflection of the scene about the yz plane. The length of vectors is preserved, as is the angle between any two vectors. But the matrix is not a rotation.

Another orthonormal matrix that fails the "$\det \mathbf{R} = 1$" condition is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which swaps the x and y variables. Swapping yz or xz also fails.

In the above counterexamples, we are reflecting the scene or swapping two coordinates. Essentially this changes objects from a right to a left handed coordinate system.

[1] You may recall from your linear algebra background that such matrices are special cases of *unitary* matrices (which can have complex elements and can have determinant -1). If you wish to read more about rotations, see e.g. `http://en.wikipedia.org/wiki/Rotation_group`.

Example 1

Suppose we wish to find a rotation matrix that maps the unit z vector to some given vector $\mathbf{p}$ which has unit length. i.e.

$$\mathbf{p} \;=\; \mathbf{R} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We suppose that $\mathbf{p} \neq \hat{\mathbf{z}}$, since the problem would be trivial in this case.
Solution: By inspection, the 3rd column of $\mathbf{R}$ has to be $\mathbf{p}$ itself. Moreover, in order for $\mathbf{R}$ to be a rotation matrix, it is necessary that the first two columns of $\mathbf{R}$ are perpendicular to the vector $\mathbf{p}$, that is, to the third column. What might these first two columns of $\mathbf{R}$ be ?

Since $\mathbf{p} \neq \hat{\mathbf{z}}$, we consider the vector

$$\mathbf{p}' = \frac{\mathbf{p} \times \hat{\mathbf{z}}}{\| \mathbf{p} \times \hat{\mathbf{z}} \|}$$

and note that $\mathbf{p}'$ is perpendicular to $\mathbf{p}$. Hence we can use it as one of the columns of our matrix $\mathbf{R}$. (More generally, note $\mathbf{p}'$ it is perpendicular to the plane spanned by $\mathbf{p}$ and $\mathbf{z}$.)

We now want a third vector, which is perpendicular to both $\mathbf{p}$ and **p'**. We chose $\mathbf{p}' \times \mathbf{p}$, noting that it is of unit length, since **p'** is perpendicular to $\mathbf{p}$ and both of the latter are of unit length.

We want to build a matrix $\mathbf{R}$ which preserves handedness. In particular, since $\hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{z}}$, we want that $\mathbf{R}\hat{\mathbf{x}} = \mathbf{R}\hat{\mathbf{y}} \times \mathbf{R}\hat{\mathbf{z}}$. How do we do it? Recall $\mathbf{p} = \mathbf{R}\hat{\mathbf{z}}$, so $\mathbf{p}$ is in the third column of $\mathbf{R}$. By inspection, if were to put $\mathbf{p}'$ in the second column of $\mathbf{R}$, we would have

$$\mathbf{R}\hat{\mathbf{y}} = \mathbf{p}'$$

and similarly if we were to put $\mathbf{p}' \times \mathbf{p}$ in the first column of $\mathbf{R}$, we would have

$$\mathbf{R}\hat{\mathbf{x}} = \mathbf{p}' \times \mathbf{p}.$$

So

$$\mathbf{R}\hat{\mathbf{x}} = \mathbf{p}' \times \mathbf{p} = \mathbf{R}\hat{\mathbf{y}} \times \mathbf{R}\hat{\mathbf{z}}$$

which was what we wanted. i.e. to preserve the handedness of the axes. Thus, twe could let the three columns of $\mathbf{R}$ be defined:

$$\mathbf{R} = [\mathbf{p}' \times \mathbf{p} \;,\; \mathbf{p}' \;,\; \mathbf{p}].$$

Example 2

Suppose we wished to have a rotation matrix that rotates a given unit length vector $\mathbf{p}$ to the z axis, i.e.

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \;=\; \mathbf{R}\,\mathbf{p}.$$

Obviously this can be done by taking the transpose of the solution of Example 1, since the problem just asks for the inverse of the problem of Example 1. But let's look at a more direct solution. (Note: to save time, I didn't go through the following in class.)

By similar reasoning to Example 1, we can see that the 3rd row of the rotation matrix has to be $\mathbf{p}$ itself. Again, for $\mathbf{R}$ to be a rotation matrix, it is necessary that the first two rows of $\mathbf{R}$ are perpendicular to third row. The first two rows could again be $\mathbf{p}' \times \mathbf{p}$ and $\mathbf{p}'$, respectively. The rows of $\mathbf{R}$ in this example are the new coordinate axes for the scene. These new coordinate axes are righthanded, since trivially $(\mathbf{p}' \times \mathbf{p}) = \mathbf{p}' \times \mathbf{p}$, i.e. first row equals second row cross third row.

Notice that in both Examples 1 and 2, there are many possible solutions. For Example 1, we can perform any rotation $\mathbf{R}_z$ about the z axis prior to multiplying by the solution $\mathbf{R}$ and we will still have a solution. That is, if $\mathbf{R}$ is a solution for Example 1, then so is $\mathbf{R}\ \mathbf{R}_z(\theta)$ for any θ. Similarly, for Example 2, we can provide any rotation about the $\hat{z}$ axis after $\mathbf{R}$, that is, if $\mathbf{R}$ is a solution for Example 2, then so is $\mathbf{R}_z(\theta)\mathbf{R}$ for any θ.

Axis of rotation

You may be finding it confusing to use the term "rotation" for these matrices. One thinks intuitively of rotation as a continuous operation - not a single discrete mapping. In particular, the notion of an axis of rotation is meaningful when we think of a spinning body such as the Earth or a baseball. But it may not be obvious what we mean by the rotation axis for these matrices. So how should we think about the axis of rotation.

From linear algebra, you can verify that a 3D rotation matrix (as defined earlier) must have at least one real eigenvalue and because rotation matrices preserve vector lengths, this eigenvalue must be 1. The eigenvector $\mathbf{p}$ that corresponds to this eigenvalue satisfies

$$\mathbf{p} = \mathbf{R}\mathbf{p} \ .$$

Note that this vector doesn't change under the rotation. This vector is thus the axis of rotation defined by the matrix $\mathbf{R}$.

Example 3

Let's now come at the problem from the other direction. Suppose we wish to define a rotation matrix that rotates by an angle θ about a given axis $\mathbf{p}$. Obviously if $\mathbf{p}$ is one of the canonical axes $(\hat{\mathbf{x}}, \hat{\mathbf{y}}$, or $\hat{\mathbf{z}})$, then it is easy. So, let's assume that $\mathbf{p}$ is not one of the canonical axes.

One easy way to do it is to first rotate $\mathbf{p}$ to one of the canonical axes (Example 2), then perform the rotation by θ about this canonical axis, and then rotate the canonical axis back to $\mathbf{p}$ (Example 1). Symbolically,

$$\mathbf{R}_{z\rightarrow p}\ \mathbf{R}_z(\theta)\ \mathbf{R}_{p\rightarrow z} \ = \ \mathbf{R}_p(\theta).$$