1. (a)

$$\begin{aligned}\mathbf{p}(s,t) &= (s, y_0, t)\\ \mathbf{n}(s,t) &= (0,1,0)\\ \mathbf{p}_{bump}(s,t) &= \mathbf{p}(s,t) + b(s,t)(0,1,0)\\ &= (s, y_0 + b(s,t), t)\end{aligned}$$

(b)

$$\begin{aligned}\frac{\partial \mathbf{p}_{bump}(s,t)}{\partial s} &= \frac{\partial \mathbf{p}(s,t)}{\partial s} + \frac{\partial b(s,t)}{\partial s}\mathbf{n}(s,t)\\ &= (1,0,0) + \frac{\partial b(s,t)}{\partial s}(0,1,0)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \mathbf{p}_{bump}(s,t)}{\partial t} &= \frac{\partial \mathbf{p}(s,t)}{\partial t} + \frac{\partial b(s,t)}{\partial t}\mathbf{n}(s,t)\\ &= (0,0,1) + \frac{\partial b(s,t)}{\partial t}(0,1,0)\end{aligned}$$

So,

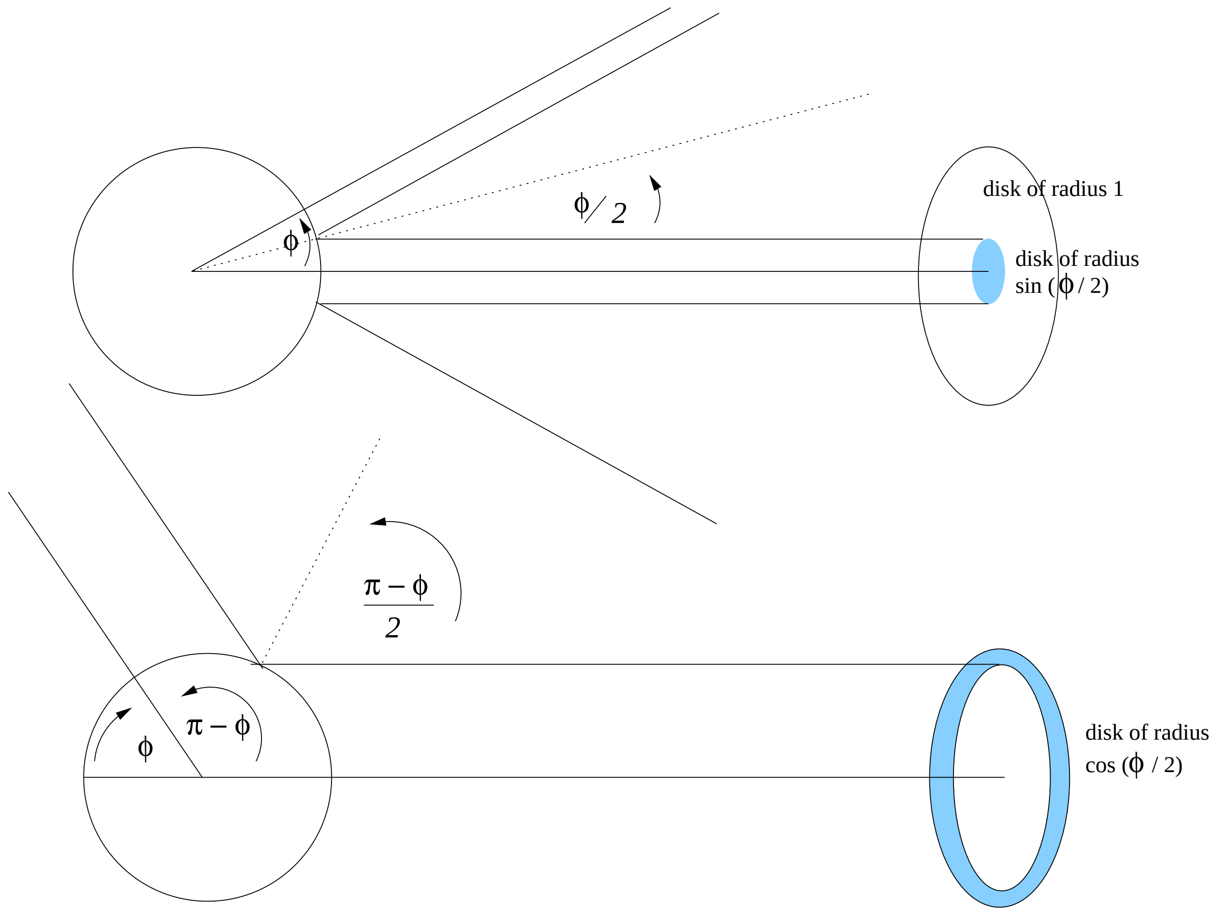
$$\frac{\partial \mathbf{p}_{bump}(s,t)}{\partial s} \times \frac{\partial \mathbf{p}_{bump}(s,t)}{\partial t} = (0,-1,0) + \frac{\partial b(s,t)}{\partial s}(1,0,0) + \frac{\partial b(s,t)}{\partial t}(0,0,1)$$

This vector is in the direction of the surface normal. To get a unit surface normal you need to normalize.

2. Let's deal with the general case first. Consider a cone of radius ϕ centered in the viewing direction. For a ray that is parallel to this cone to be reflected in the direction of the z axis, this ray must strike the mirror sphere at a point that is $\frac{\phi}{2}$ degrees from the z axis direction. Thus the set of directions in the scene that are bounded by this cone and that reflect in the z axis direction will be represented on the disk map by a disk of radius $\sin(\frac{\phi}{2})$. This disk has area $\pi \sin(\frac{\phi}{2})^2$.

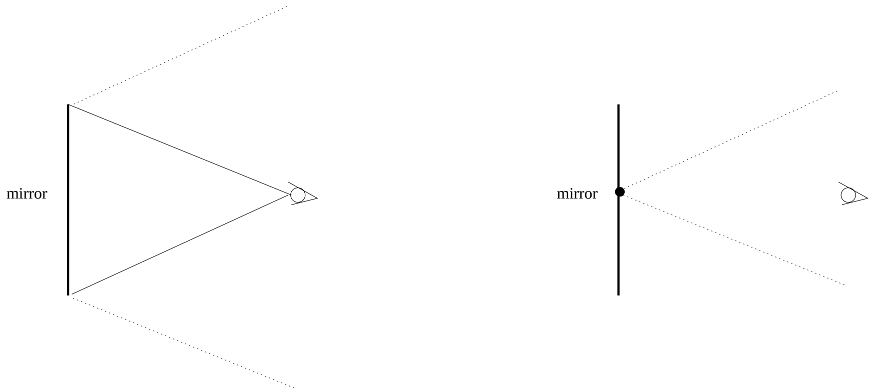
   By a similar argument, the environment map directions that are on the far side of the sphere and lie in a cone of radius $\pi - \phi$ that is centered in the opposite direction $(-z)$ will be reflected in direction z when they reflect off an annulus of points on the disk map. This annulus is the whole disk of unit radius minus a disk of radius $\sin(\frac{\pi-\phi}{2}) = \cos(\frac{\phi}{2})$. The area of this annulus is $\pi - \pi\cos(\frac{\phi}{2})^2 = \pi\sin(\frac{\phi}{2})^2$ which is the same as the area above.

   Note that the first part of the question is a special case where $\phi = \frac{\pi}{2}$.



ϕ
ϕ/2
disk of radius 1
disk of radius
sin (ϕ / 2)
π − ϕ
2
π − ϕ
ϕ
disk of radius
cos (ϕ / 2)

3. The view volume that would be visible in the two cases lies between the dotted lines. The case on the left is ray tracing. The case on the right is environment mapping (with environment map computed from the black dot).



4. In the following derivation, any expression that does *not* have an α subscript really should have a $rgb\alpha$ subscript. For example, (F over M) refers to (F over $M)_{rgb\alpha}$. I leave out these extra subscripts to keep it simple. See me if you don't understand.

$$\begin{aligned}
& (F \text{ over } M) \text{ over } B \\
= \; & (F + (1 - F_\alpha)M) \text{ over } B \\
= \; & (F + (1 - F_\alpha)M) \; + (1 - (F + (1 - F_\alpha)M)_\alpha) \; B \\
= \; & F + (1 - F_\alpha)M \; + (1 - (F_\alpha + (1 - F_\alpha)M_\alpha)) \; B \\
= \; & F + (1 - F_\alpha)M \; + (1 - F_\alpha)B - (1 - F_\alpha)M_\alpha \; B \\
= \; & F + (1 - F_\alpha)M \; + (1 - F_\alpha)(1 - M_\alpha) \; B
\end{aligned}$$

Similarly,

$$\begin{aligned}
& F \text{ over } (M \text{ over } B) \\
& F \; + \; (M \; + (1 - M_\alpha) \; B) \\
& F \; + \; (1 - F_\alpha)(M \; + (1 - M_\alpha) \; B) \\
= \; & F + (1 - F_\alpha)M \; + (1 - F_\alpha)(1 - M_\alpha) \; B
\end{aligned}$$

5. The points on a surface are parametrized by two variables (s, t). For each point, the illumination reaching the point can come from a hemisphere of directions and a hemisphere is parametrized by two variables e.g. (ϕ, ψ). Thus, in general, illumination at a surface point is a function of four parameters $I_{illumination}(s, t, \phi, \psi)$.

   In environment mapping, we approximate the illumination by assuming it does *not* depend on s, t, so $I_{illumination}(s, t, \phi, \psi) = E(\phi, \psi)$. However, in general illumination *can* depend on s, t as well as ϕ, ψ.

6. $g(x,y,z) = (1-x)\ g(0,y,z) + x\ g(1,y,z)$

$$= (1-x)((1-y)g(0,0,z) + yg(0,1,z)) + x((1-y)g(1,0,z) + y\ g(1,1,z))$$

$$\begin{aligned} = \ & (1-x)\ (1-y)\ (1-z)\ g(0,0,0) + \\ & (1-x)\ (1-y)\ z\ g(0,0,1) + \\ & (1-x)\ y\ (1-z)\ g(0,1,0) + \\ & (1-x)\ y\ z\ g(0,1,1) + \\ & x\ (1-y)\ (1-z)\ g(1,0,0) + \\ & x\ (1-y)\ z\ g(1,0,1) + \\ & x\ y\ (1-z)\ g(1,1,0) + \\ & x\ y\ z\ g(1,1,1) \end{aligned}$$

Notice that there is one term in the sum for each of the eight corners of the cube, and the weight factor for each corner is determined by whether the parameter (x,y,z) is 0 or 1.

7. Let the strength of the emitted spectra at a pixel be $\mathbf{s}_{rgb}$, so the emitted spectra can be expressed $\mathbf{Ds}$. The reflected ambient light contributes a spectrum $a(\lambda)$ at each pixel. Thus, the LMS values are:

$$\begin{bmatrix} I_L \\ I_M \\ I_S \end{bmatrix} = \mathbf{Ca} + \mathbf{CDs}$$

Notice that the ambient component of light reflected from the monitor leads to a translation in LMS space. (Homogeneous coordinates could be used for this situation.)

8. The statement is true.

If a person is missing the L cones, then the color sensitivity matrix $\mathbf{C}$ has only two rows instead of three. In this case, two spectra $I_1(\lambda)$ and $I_2(\lambda)$ are indistinguishable (metameric) for this person if and only if

$$\begin{bmatrix} C_M \\ C_S \end{bmatrix}_{2\times N} I_1(\lambda) = \begin{bmatrix} C_M \\ C_S \end{bmatrix}_{2\times N} I_2(\lambda)$$

If a person missing the L cones can discriminate the two, then the pairs are not the same. In this case, it trivially follows that

$$\begin{bmatrix} C_L \\ C_M \\ C_S \end{bmatrix}_{3\times N} I_1(\lambda) \neq \begin{bmatrix} C_L \\ C_M \\ C_S \end{bmatrix}_{3\times N} I_2(\lambda)$$

and so a person with all three types of cones can discriminate the two spectra also.

Notice that the converse does not hold. There may be spectrum pairs that a person with all three cones can discriminate, but a person with only two cone systems cannot discriminate. (And that is what we mean by color blindness.)

9. Using the formula $(F \text{ over } B)_\alpha = F_\alpha + (1 - F_\alpha)B_\alpha$, we get

$$(F \text{ over } B)_\alpha = (1, \frac{13}{16}, \frac{3}{4}, \frac{13}{16}, 1).$$

10. We have

$$(F \text{ over } B)_b = F_b + (1 - F_\alpha)B_b = (1 - F_\alpha)$$

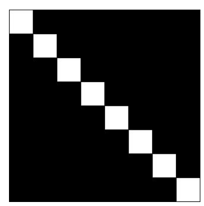
since $B_b = 1$ and $F_b = 0$. The left side is measured, so we immediately get F_α. This is good enough since

$$(F \text{ over } B)_{rg} = F_{rg} + (1 - F_\alpha)B_{rg} = F_{rg}$$

since $B_{rg} = 0$, so we can get F_r and F_g as well.

11.

In this case, $s = 2$ (we're scaling x, y by a factor of $\frac{1}{2}$) and $N = 2$ (we're getting twice as many squares). So when we plug this into the formula we get: $\frac{\log 2}{\log 2} = 1$, which is the dimensionality of a line.



12. We calculate $\mathbf{r}$ using the formula for mirror reflection given in class.

$$\mathbf{l} + \mathbf{r} = 2(\mathbf{n} \cdot \mathbf{l})\mathbf{n} \; .$$

This formula assumes $\mathbf{l}$ has been normalized to unit length, so

$$\mathbf{l} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \; .$$

Since the ground plane is $y = 0$, the surface normal to the ground plane is $\mathbf{n} = (0,1,0)$. Thus, $\mathbf{n} \cdot \mathbf{l} = \frac{2}{3}$ which we can plug into the formula for $\mathbf{r}$. We find that the mirror reflection direction is $\mathbf{r} = (-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$.

According to the Phong model, the peak of the highlight occurs for a ray that is in direction $\mathbf{r}$. This ray must pass through the camera which is at (4,6,7). Hence this ray belongs to a line with parametric equation:

$$(x, y, z) = (4, 6, 7) + t \; (-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}) \; .$$

We need to compute where this line intersects the $y = 0$ plane. This occurs at $0 = 6 + \frac{2}{3}t$, or $t = -9$. Thus, the peak of the highlight occurs at

$$(x, y, z) \;=\; (7, 0, 13).$$

13. The specular highlight is at its peak when the angle α between $\mathbf{r}$, the mirror direction, and $\mathbf{v}$, the vector from the surface towards the camera, is equal to 0. Hence we make $\mathbf{v} = \mathbf{r}$. But how do we find what $\mathbf{r}$ is?

    We can rearrange the equation for mirror reflection so that we solve for $\mathbf{r}$:

$$\mathbf{r} = 2(\mathbf{n} \cdot \mathbf{l})\mathbf{n} - \mathbf{l}.$$

    From the plane equation we get the normalized plane normal to be $\mathbf{n} = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$. Plugging this into the above formula gives us $\mathbf{r} = (\frac{4}{9}, \frac{-7}{9}, \frac{4}{9})$.

    The peak of the highlight occurs from a ray that is in direction $\mathbf{r}$. This ray must pass through the viewer who is at $(0, 0, 0)$. Hence this ray belongs to a line with parametric equation:

$$(x, y, z) = (0, 0, 0) + t(\frac{4}{9}, \frac{-7}{9}, \frac{4}{9}).$$

    We are interested in where this line intersects the plane $2x + y + 2z + 17 = 0$. This occurs at $t = -17$. Thus the point of this peak highlight ray occurs at:

$$(x, y, z) = (-\frac{68}{9}, \frac{119}{9}, -\frac{68}{9})$$

14. The formula we use is:

$$I(x, y) = I(x_0, y_0) + (I(x_1, y_1) - I(x_0, y_0))(\frac{x - x_0}{x_1 - x_0})$$

    We will need to interpolate three times. First we interpolate between I_1 and I_2:

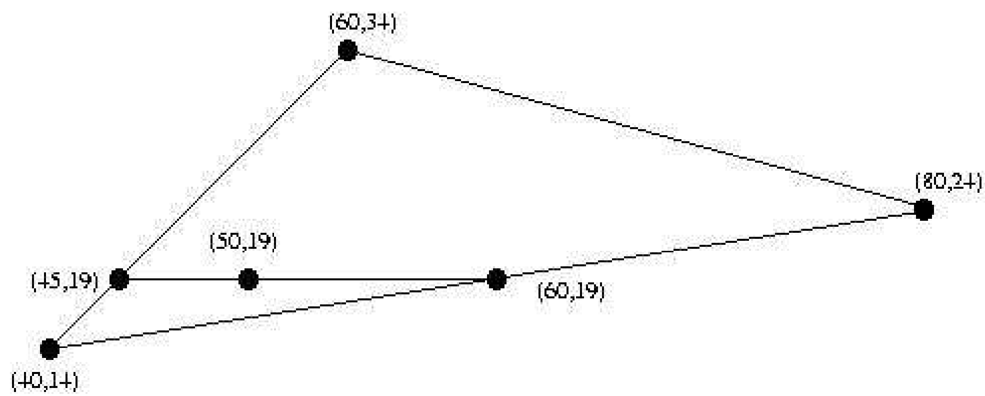
$$I(45, 19) = 10 + (90 - 10)(\frac{45 - 40}{60 - 40}) = 10 + 20 = 30$$

    Next we interpolate between I_1 and I_3:

$$I(60, 19) = 10 + (140 - 10)(\frac{60 - 40}{80 - 40}) = 10 + 65 = 75$$

    Finally, we interpolate between $I(45, 19)$ and $I(60, 19)$:

$$I(50, 19) = 30 + (75 - 30)(\frac{50 - 45}{60 - 45}) = 30 + 15 = 45$$


(60,34)
(80,24)
(50,19)
(45,19)
(60,19)
(40,14)