

1. For each non-trivial case, you would need to write the matrices in their general form, and then multiply them through. For example,  $\mathbf{TR} \neq \mathbf{RT}$ :

$$\begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & T_x \\ R_{21} & R_{22} & R_{23} & T_y \\ R_{31} & R_{32} & R_{33} & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

whereas

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & (R_{11}T_x + R_{12}T_y + R_{13}T_z) \\ R_{21} & R_{22} & R_{23} & (R_{21}T_x + R_{22}T_y + R_{23}T_z) \\ R_{31} & R_{32} & R_{33} & (R_{31}T_x + R_{32}T_y + R_{33}T_z) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. For a unit sphere  $x^2 + y^2 + z^2 = 1$ . Defining

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

we can write the equation of a sphere as:

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

The ellipsoid described in the question is obtained by transforming a unit sphere. The transformation is a scaling, followed by rotation, followed by translation:

$$\begin{bmatrix} wx' \\ wy' \\ wz' \\ w \end{bmatrix} = \mathbf{T R S} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Note that the rotation matrix is formed by putting the desired  $(1, 1, 0)^T$  into the third column (after making it of unit length), and then choosing the first two columns so that you have an orthonormal basis of  $\mathbb{R}^3$ . (Here I am ignoring the fourth i.e. homogeneous coordinate, since it plays no role in rotation.)

To obtain the equation of the ellipse, you need to do a substitution. You invert the above equation to isolate  $(x, y, z, 1)$ ,

$$(\mathbf{T R S})^{-1} \begin{bmatrix} wx' \\ wy' \\ wz' \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

and then substitute into the equation for a sphere, i.e.

$$\begin{bmatrix} wx' & wy' & wz' & w \end{bmatrix} (\mathbf{T R S})^{-T} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} (\mathbf{T R S})^{-1} \begin{bmatrix} wx' \\ wy' \\ wz' \\ w \end{bmatrix} = 0.$$

3. To transform from world coordinates to camera coordinates, we translate by  $(-3, -4, -3)$  and then rotate onto the vectors  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{n}$ . Projecting orthographically means by throwing away the  $\mathbf{n}$  coordinate.

To compute the transformations, use:

- the normal  $\mathbf{N}$  to a plane  $Ax + By + Cz = D$  is in the direction  $(A, B, C)$ .
- for a right handed coordinate system, the  $\mathbf{u}$  and  $\mathbf{v}$  vectors defining the view reference coordinates are in directions  $\mathbf{VUP} \times \mathbf{N}$  and  $\mathbf{N} \times (\mathbf{VUP} \times \mathbf{N})$ , respectively.

The unit vectors  $\mathbf{n}$ ,  $\mathbf{u}$ ,  $\mathbf{v}$  are:

$$\mathbf{n} = \left( \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right), \quad \mathbf{u} = \left( -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0 \right), \quad \mathbf{v} = \left( \frac{1}{\sqrt{30}}, -\frac{2}{\sqrt{30}}, -\frac{5}{\sqrt{30}} \right)$$

The desired transformation can be written as:

$$\begin{bmatrix} u \\ v \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 & 0 \\ \frac{1}{\sqrt{30}} & \frac{-2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

4. (a)

$$\mathbf{C}_{3 \times 2} = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix}$$

$$[\mathbf{q}(0) \quad \mathbf{q}(1) \quad \mathbf{q}(2)] = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

so

$$[\mathbf{q}(0) \quad \mathbf{q}(1) \quad \mathbf{q}(2)] \begin{bmatrix} 0 & 1 & 4 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix}$$

(b)

$$\mathbf{q}'(t) = \mathbf{C} \begin{bmatrix} 2t \\ 1 \\ 0 \end{bmatrix}$$

(c) We can define a unique curve through two given 2D points by using the tangent vector of only one of the points, say  $\mathbf{q}'(0)$ , as follows:

$$\mathbf{q}(t) = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$$

and so

$$[\mathbf{q}(0) \quad \mathbf{q}(1) \quad \mathbf{q}'(0)] = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

so

$$[\mathbf{q}(0) \quad \mathbf{q}(1) \quad \mathbf{q}'(0)] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix}$$

Thus we would define the curve by:

$$\mathbf{q}(t) = [\mathbf{q}(0) \quad \mathbf{q}(1) \quad \mathbf{q}'(0)] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$$