

1.

$$\mathbf{p} \times \mathbf{q} = (13, 13, -13) \qquad \mathbf{p} \cdot \mathbf{q} = 39$$

2.

$$\mathbf{p}' = (2, 7, 4, 3) = \left(\frac{2}{3}, \frac{7}{3}, \frac{4}{3}, 1\right) \qquad \mathbf{q}' = (1, 3, 5, 2) = \left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 1\right)$$

so we need to calculate the cross and dot products of

$$\left(\frac{2}{3}, \frac{7}{3}, \frac{4}{3}\right) \qquad \left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right)$$

Straightforward calculation yields

$$\mathbf{p}' \times \mathbf{q}' = \left(\frac{23}{6}, -1, -\frac{1}{6}\right) \qquad \mathbf{p}' \cdot \mathbf{q}' = \frac{43}{6}$$

3. (a) Translate the scene so that the point $(4, -1, 1)$ maps to the origin, then scale the scene in the y direction, then translate the origin back to the point $(4, -1, 1)$.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- (b) First rotate to one of the canonical axes (I choose the z -axis), then perform the rotation about the canonical axis, and then rotate the canonical axis back (this can be done by the inverse or transpose of our first rotation matrix).

First, we look at the rotation matrix that rotates the vector $p = (1, 1, 0)$ to the z -axis. The third row of \mathbf{R} is just p itself, and the first and second rows are perpendicular to p and to each other, and also preserve righthandedness.

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Next, we give the matrix that rotates θ degrees clockwise around the z -axis. For example, rotating by 90 degrees would take the x axis to the y axis.

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Given these matrices, our overall transformation is give by $\mathbf{R}^T \mathbf{R}_z(\theta) \mathbf{R}$

4. There are several possible solutions. For example,

$$\mathbf{TRS} = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{TSR} = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

5. Notice

$$(x, y, z, 1) \rightarrow (x, y, z, 2y) \equiv \left(\frac{x}{2y}, \frac{1}{2}, \frac{z}{2y}, 1 \right)$$

and so we are mapping to the $y = \frac{1}{2}$ plane.

6. The following solution first does the translation and then does the scaling.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{f_1 - f_0} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -f_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7. In class, I derived a projection matrix that keeps f_0 and f_1 constant,

$$\mathbf{M}_0 = \begin{bmatrix} f_0 & 0 & 0 & 0 \\ 0 & f_0 & 0 & 0 \\ 0 & 0 & f_0 + f_1 & -f_0 f_1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We now wish to normalize this coordinate system, so that the view volume maps to $(x, y, z) \in [-1, 1] \times [-1, 1] \times [-1, 1]$. We do this in a few steps. First, we translate in the z direction to bring $z = [f_0, f_1]$ to $z = [0, f_1 - f_0]$. Then we scale each of the x, y, z dimensions. (The “-” in the x and y scaling factors are needed because $f_0 < 0$.)

$$\mathbf{M}_1 = \begin{bmatrix} \frac{-1}{f_0 \tan(\frac{\theta}{2})} & 0 & 0 & 0 \\ 0 & \frac{-1}{f_0 \tan(\frac{\theta}{2})} & 0 & 0 \\ 0 & 0 & \frac{1}{f_0 - f_1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -f_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, we need to normalize the view volume. We remap the z coordinate to take $[-1, 0]$ to $[-1, 1]$. We do so by scaling by 2 in the z direction, then translating by 1 in the z direction. As a last step, we reflect about the $z = 0$ plane (see lecture 6). This yields the matrix product:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{M}_1 \mathbf{M}_0.$$

In lecture 6, I mentioned one subtlety which is that OpenGL multiplies the \mathbf{M}_0 matrix by -1, so as to avoid funny stuff that can happen when clipping. You can either multiply the \mathbf{M}_0 matrix by -1, or just multiply the final transformation by -1, since you get the same result.

If you don't believe me, then try it yourself by inserting the following lines in an appropriate place in your code.

```
GLfloat m[16];

glMatrixMode(GL_PROJECTION);
glLoadIdentity();
gluPerspective( 90.0, 1.0 , 4.0 , 20.0 );

glGetFloatv(GL_PROJECTION_MATRIX,m);
// OpenGL stores projections in row major order.
for (i = 0; i <= 3; i++) {
    printf("%f %f %f %f\n", m[i], m[4+i], m[8+i], m[12+i]);
}
```

8. Since the `aspect` parameter is 1, we have

$$\text{left} = \text{bottom} = -\text{right} = -\text{top}.$$

Since the `fov` is 60 degrees, the angle from the center of the image to the edge of the image (along either the x or y axis) is 30 degrees. The `near` plane is at a distance 4. Using the hint and the fact that the parameters `left`, `bottom`, `right`, `top` are defined by position in the `near` plane, we get that the first four arguments are: hint and the fact that the parameters

$$(\text{left}, \text{right}, \text{bottom}, \text{top}) = \left(-\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}, -\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$$