Lecture 13

review of Exercises 2-11
2. We say that two matrices $A$ and $B$ commute if $A B = B A$. Matrices do not commute in general, but certain types of matrices do. Which of the following types of matrices commute? Assume matrices are $4 \times 4$.

- scaling matrices $S_1 S_2$
- translation matrices $T_1 T_2$
- rotation matrices, $R_1 R_2$
- rotation and translation, $RT$
- scaling and translation, $ST$
- scaling and rotation, $SR$
But note... scaling and translation sometimes do commute.
1. Write the following as a sequence of $4 \times 4$ matrices. Scale the scene by a factor 3 in the $y$ direction, while leaving the position of the point $(x, y, z) = (4, -1, 1)$ unchanged.

\[ \bullet (4, -1, 1) \]
solution 1:

\[
\begin{bmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & -4 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

solution 2:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Variation on the problem: rotate the scene about the point \((4, 1, -1)\) and about some axis.
5. One can define an ellipsoid surface in \( \mathbb{R}^3 \) by scaling, rotating, and translating a unit sphere. Define an ellipsoid whose:

- center is \((8, 1, 3)\) in world coordinates,
- \(x, y, z\) axes are of length 1, 1, 4, respectively.
- \(z\) axis is in direction \((1, 1, 0)\) in world coordinates,

For your answer, it is sufficient that you specify suitable matrices \(S, R, T\), and then use them to define a \(4 \times 4\) quadric matrix \(Q\).
unit sphere $\vec{x}$

ellipsoid $\frac{x'}{x}$

\[
\vec{x'} = T R S \vec{x}
\]

\[
\begin{bmatrix}
  x' \\
  y' \\
  z'
\end{bmatrix} = T R S 
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 8 \\
  0 & 1 & 0 & 1 \\
  0 & 0 & 1 & 3 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
  0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 4 & 0 \\
  0 & 0 & 0 & 1 \\
  1
\end{bmatrix}
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 4 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}
\]
We have the equation of a unit sphere.

For a unit sphere \( x^2 + y^2 + z^2 = 1 \),

\[
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix} = 0
\]

and so

\[
Q = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
  x \\
  Q
\end{bmatrix}
= 0
\]
\[(TRS)^{-1} \frac{\partial}{\partial x} = \frac{\partial}{\partial x}\]

\[
\frac{\partial^2}{\partial x^2} + Q \frac{\partial^2}{\partial x^2} = 0
\]

Substitute

Unit sphere \( \frac{\partial}{\partial x} \)

Ellipsoid \( \frac{\partial}{\partial x} \)
\[ \overrightarrow{x}' \quad (\text{TRS})^{-T}Q(\text{TRS})^{-1} \overrightarrow{x}' = 0 \]

Unit sphere \( \overrightarrow{x} \)

Ellipsoid \( \overrightarrow{x}' \)
Exercises 4  Q2 and Q3

2. Suppose the eye is at \((0, 0, -f)\) where \(f < 0\) and the projection plane is \(z = 0\). Also suppose the camera’s \(y\) and \(z\) axes are the same as those axes for world coordinates. What is the projection matrix?

3. For the previous question, what happens when \(f \to -\infty\)?
Translation by $f$ gives the situation presented in class. Note $f < 0$. 
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -f \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
f & 0 & 0 & 0 \\
0 & f & 0 & 0 \\
0 & 0 & f & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & f \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
f & 0 & 0 & 0 \\
0 & f & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & f
\end{bmatrix},
\]

translate by \(-f\)  
projection given in lecture  
(to origin)  
translate by \(f\)
Q: What happens if f goes to -infinity?
A: orthographic projection

\[
\begin{bmatrix}
  f & 0 & 0 & 0 \\
  0 & f & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 1 & f \\
\end{bmatrix}
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & \frac{1}{f} & 1 \\
\end{bmatrix}
\]

When \( f \) goes to -infinity, \( 1/f \) goes to 0.
2. Give the $4 \times 4$ projective transformation matrix that is defined by

\[ \text{gluPerspective}(\theta, \text{aspect}, \text{near}, \text{far}). \]

Rather than just plugging in the values given in the lecture, think about the special constraints on \text{gluPerspective}, as opposed to \text{glFrustum}.
\[
M_{\text{projective}} = \begin{bmatrix}
\text{near} & 0 & 0 & 0 \\
0 & \text{near} & 0 & 0 \\
0 & 0 & \text{near} + \text{far} & \text{near} \ast \text{far} \\
0 & 0 & -1 & 0
\end{bmatrix}.
\]
glFrustum( left, right, bottom, top, near, far)

For gluPerspective:

\[
\begin{align*}
\text{left} &= \ - \text{right} \\
\text{bottom} &= \ - \text{top}
\end{align*}
\]

These are defined in the plane: \( z = -\text{near} \)
$M_{\text{normalize}} = T_z(1) \cdot S \cdot T_z(\text{near})$
$$M_{\text{normalize}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\text{near} \cdot \text{aspect} \cdot 2 \cdot \tan\left(\frac{\theta}{2}\right)} & 0 & 0 & 0 \\ 0 & \frac{2}{\text{near} \cdot 2 \cdot \tan\left(\frac{\theta}{2}\right)} & 0 & 0 \\ 0 & 0 & \frac{2}{\text{near} \cdot \text{far}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \text{near} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_z(-1) \quad S \quad T_z(\text{near})$$
2. Suppose you have a line segment joining two points \((a, b, c, d)\) and \((a', b', c', d')\) in clip coordinates and you wish to clip the line segment to a boundary plane, say \(x = 1\) of the normalized view volume. Show how you can do this directly in clip coordinates. You do not need to do perspective division to find the intersection.
\[(a, b, c, d) \rightarrow \chi = \begin{bmatrix} a' & b' & c' & d' \end{bmatrix}
\]

\[(a, b, c, d) + (a', b', c', d')(1-t)\]
\[(a, b, c, d) + (a', b', c', d') (1-t) = (w, *, *, *, w)\]

Two equations, two unknowns. Solve for \( t \).
Solve for $t$.\[ t = \frac{a - d'}{(a - a') - (d - d')} \] need to do this for $x = \pm 1, y = \pm 1, z = \pm 1$ i.e. six divisions
In the pipeline lecture (7), I did not mention "perspective division" (clip coordinates to normalized device coordinates) and the mapping from NDC to window coordinates. Be aware that this mapping does need to be computed before rasterization.
Perspective division also requires six divisions.

\[
\left( \frac{a}{d}, \frac{b}{d}, \frac{c}{d} \right) \quad \left( \frac{a'}{d'}, \frac{b'}{d'}, \frac{c'}{d'} \right)
\]
4. How do you do line clipping if one of the vertices has \( w < 0 \)? Are any special considerations required or can you just go ahead and use the method of Q2 above?

Hint: there are issues. What are they? Look at the ABCD...JK figure from lecture 5 and think about what happens if you clip.
Camera coordinates

(z = f₀ + f₁)

(z = f₁)

(z = f₀)

(z = 0)

(non-normalized) projective coordinates
2. The depth buffer partitions the near to far range of depth into intervals using fixed precision (typically 24 bits, hence $2^{24}$ intervals). The mapping from continuous to discrete depth happens late in the pipeline, in particular, it maps from the normalized device depth coordinate to a (24 coding) of the range $[0,1]$.

The depth ”bins” do not correspond to uniform sized depth bins in the original scene space, however. Why not? Are the close bins bigger or smaller than the far bins, when measured in the original scene space?
\[ M_{\text{projective}} = \begin{bmatrix} \text{near} & 0 & 0 & 0 \\ 0 & \text{near} & 0 & 0 \\ 0 & 0 & \text{near + far} & \text{near * far} \\ 0 & 0 & -1 & 0 \end{bmatrix} \]
\[ z = f_0 + f_1 - \frac{f_0 f_1}{z} \]
Exercises 8 Q6

eye

Diagram with three triangles and four shapes.
6. One problem with the hierarchical bounding volume algorithm discussed in class is that there is no attempt to visit the children of a node (volume) in an optimal order. This is a problem because you easily expand a subtree that turns out to be entirely hidden.

An alternative approach is to traverse the tree by only expanding a node (bounding volumes) when the intersection with this node is closer than the intersection of other "candidate" nodes. This can be done by maintaining a separate data structure of nodes – a priority queue of nodes – whose keys are the distances to the node. How would this work? Write out such an algorithm.

Smart HBV algorithm would expand this one first.
eye

A

priority queue

\text{min} \rightarrow \square \rightarrow \bullet \rightarrow \square
p = NULL
\n\nt = infinity

if ray hits root’s BV{
    insert root into heap
}

while (heap is not empty and distance to heap.peek < t) {
    curNode = remove min from heap
    if curNode is a leaf{
        intersect ray with leaf’s surface
        if (t_intersect < t)
            t = t_intersect
            p = surface
    }
    else
        for each child of curNode{
            intersect ray with child’s BV
            if t_intersect < t
                insert child into heap
        }
}
General idea: at any time, there is a queue of (candidate) bounding volumes still to be expanded.
2. Consider a Koch-like surface defined by the recursive production rule illustrated in the figure. A equilateral triangle is partitioned into 4 subtriangles, and the central one is replaced by a 3 sided pyramid. Repeating this recursive procedure infinitely many times, what is the dimension of the resulting surface?
\[ C = S^D \]

\[ S = 2 \quad C = 6 \]

\[ D = \frac{\log C}{\log S} = \frac{\log 6}{\log 2}. \quad \sim 2.58 \]
1. What are geometry and blending matrices for Catmull-Rom spline?

\[ \vec{p}(t) = \vec{G} \cdot B \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} \]
Recall scenario for Catmull-Rom spline:
\[ p_i = \frac{1}{2} (p_{i+1} - p_{i-1}) \]

\[ p_{i+1} = \frac{1}{2} (p_{i+2} - p_i) \]
$$G_{\text{Hermite}}$$

$$\begin{bmatrix}
    p(i) & p(i+1) & p'(i) & p'(i+1)
\end{bmatrix}$$

$$= \begin{bmatrix}
    p(i-1) & p(i) & p(i+1) & p(i+2)
\end{bmatrix} \begin{bmatrix}
    0 & 0 & -\frac{1}{2} & 0 \\
    1 & 0 & 0 & -\frac{1}{2} \\
    0 & 1 & \frac{1}{2} & 0 \\
    0 & 0 & 0 & \frac{1}{2}
\end{bmatrix}$$
\[ G_{Hermite} B_{Hermite} = G \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} B_{Hermite} = GB_{Catmull-Rom} \]
What happens to the mesh when we collapse $AB$ to $B$ and then $BC$ to $C$?

How many vertices, edges, and faces are lost?
One vertex and one edge are lost, but no faces.
One vertex, two edges, and one face are lost.
Announcements

Thursday: midterm

Last name A-P (Trottier 0100),
Last name Q-Z (Rutherford Physics 114)

There will be 10 questions, one from each lecture (2-11).

A2: this week for sure

My office hours this week:
   Wednesday (most of afternoon)
   Thursday AM