Questions

1. (a) Consider two Gaussian functions $G(x, \sigma_1)$ and $G(x, \sigma_2)$. Assume $\sigma_1 < \sigma_2$. Is the following DOG ON-center/OFF-surround or OFF-center/ON-surround?

$$\text{DOG}(x, \sigma_1, \sigma_2) = G(x, \sigma_1) - G(x, \sigma_2)$$

(b) Consider an array of retinal ganglion cells, each modelled as identical ON-center OFF-surround DOG function. Sketch the array of responses of these cells to the image of a bright disk on a black surround. Assume the bright disk is much smaller than the DOG. The DOG and disk are circularly symmetric, so you can sketch your response in 1D.

(c) Same as (b), but now suppose the bright disk is much larger than the DOG.

(d) Suppose you have an image $I(x, y) = \delta(x - x_0)$ which is a vertical white line at $x = x_0$ and otherwise is black. Sketch the result of the 2D convolution $\text{DOG}(x, y, \sigma_1, \sigma_2) * I(x, y)$. Be sure to label where $x = x_0$.

(e) How would your answer change if the image were a dark line on a white background?

2. We will often work with filters such as Gabor functions that are the product of two functions. Suppose we have two 1D functions $I(x)$ and $h(x)$ and we take their product. What can we say about the Fourier transform? The answer is similar to the convolution theorem, and indeed is just another version of that theorem:

$$\mathbf{F} (I(x)h(x)) = \frac{1}{N} \hat{I}(k) * \hat{h}(k).$$

or, in words, the Fourier transform of the product of two functions is the convolution of the Fourier transforms of the two functions. Note that the convolution on the right hand side is between two complex valued functions, rather than real valued functions. But the same definition of convolution applies.

Prove the above property, by taking the inverse Fourier transform of the right side and showing that it gives $I(x)h(x)$.

3. What are the Fourier transform of a cosine Gabor and a sine Gabor?

$$g_c(x) \equiv \cos(k_0 \frac{2\pi}{N} x) \ G(x, \sigma)$$

$$g_s(x) \equiv \sin(k_0 \frac{2\pi}{N} x) \ G(x, \sigma)$$

What about the Fourier transform of a complex Gabor

$$g_c(x) + ig_s(x) \equiv \cos(k_0 \frac{2\pi}{N} x) \ G(x, \sigma) + i \sin(k_0 \frac{2\pi}{N} x) \ G(x, \sigma)$$
Solutions

1. (a) ON-center, OFF-surround

(b) If the disk is very small, then the image can be approximated by a delta function. But a delta function convolved with any function (in particular, a DOG) is just that function itself. So, answer is a response profile that is identical to a DOG.

(c) The response is a 1D profile that looks like this:

\[
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
with period \( N \) (see page 15 of linear system notes).

\[
\mathbf{F}^{-1} \hat{I}(k) \ast \hat{h}(k) = \frac{1}{N} \sum_{k=0}^{N-1} e^{\frac{2\pi i k}{N} x} \sum_{k'=0}^{N-1} \hat{h}(k') \hat{I}(k-k')
\]

\[
= \frac{1}{N} \sum_{k'=0}^{N-1} \hat{h}(k') \sum_{k=0}^{N-1} e^{\frac{2\pi i k}{N} x} \hat{I}(k-k')
\]

\[
= \frac{1}{N} \sum_{k'=0}^{N-1} \hat{h}(k') e^{\frac{2\pi i k'}{N} x} \sum_{k=0}^{N-1} e^{\frac{2\pi i (k-k')}{N} x} \hat{I}(k-k')
\]

\[
= h(x) \sum_{k''=-k'}^{N-1-k'} e^{\frac{2\pi i k''}{N} x} \hat{I}(k''), \text{ where } k'' = k - k'
\]

\[
= Nh(x) \hat{I}(x)
\]

Thus,

\[
\frac{1}{N} \mathbf{F}^{-1} \hat{I}(k) \ast \hat{h}(k) = h(x) \hat{I}(x)
\]

and so

\[
\frac{1}{N} \hat{I}(k) \ast \hat{h}(k) = \mathbf{F} h(x) \hat{I}(x)
\]

3. We have seen in lecture 5 that the Fourier transform of \( \cos(k_0 \frac{2\pi}{N} x) \) is:

\[
\mathbf{F} \cos(k_0 \frac{2\pi}{N} x) = \frac{N}{2} (\delta(k - k_0) + \delta(k + k_0))
\]

Applying the convolution theorem from Q1 to the cosine Gabor gives:

\[
\hat{g}_c(k) = \frac{1}{N} \sum_{k'=0}^{N-1} \left( \delta(k - k_0) + \delta(k + k_0) \right) e^{-\frac{1}{2} \left( \frac{2\pi \sigma}{N} k \right)^2}
\]

\[
= \frac{1}{N} \left( e^{-\frac{1}{2} \left( \frac{2\pi \sigma}{N} (k-k_0) \right)^2} + e^{-\frac{1}{2} \left( \frac{2\pi \sigma}{N} (k+k_0) \right)^2} \right)
\]

Thus, the Fourier transform of a cosine Gabor is a pair of Gaussians, centered at the frequency of the cosine.

Similarly for the sine Gabor,

\[
\mathbf{F} \sin(k_0 \frac{2\pi}{N} x) = -i \frac{N}{2} (\delta(k - k_0) + \delta(k + k_0))
\]

and so

\[
\hat{g}_s(k) = -\frac{1}{N} i \frac{N}{2} \left( \delta(k - k_0) - \delta(k + k_0) \right) e^{-\frac{1}{2} \left( \frac{2\pi \sigma}{N} k \right)^2}
\]

\[
= -\frac{i}{2} \left( e^{-\frac{1}{2} \left( \frac{2\pi \sigma}{N} (k-k_0) \right)^2} - e^{-\frac{1}{2} \left( \frac{2\pi \sigma}{N} (k+k_0) \right)^2} \right)
\]
The Fourier transform of the complex Gabor is

\[ F(g_c(x) + ig_s(x)) = F \left( e^{i k_0 \frac{2 \pi}{N} x} \right) G(x, \sigma) \]

\[ = \delta(k - k_0) * e^{-\frac{1}{2} \left( \frac{2 \pi}{N} k \right)^2} \]

\[ = e^{-\frac{1}{2} \left( \frac{2 \pi}{N} (k - k_0) \right)^2} \]

Such complex Gabors can be very useful, namely we can use them to make bandpass filters, where \( k_0 \) is the center frequency of the band and \( \sigma \) can be chosen to get any desired bandwidth.