Questions

1. For a smooth noise-free image $I(x, y, t)$, the 2D normal velocity vector $\vec{v}_n$ at $(x, y, t)$ is defined to be the image velocity component in the direction parallel to the intensity gradient. Show that the normal velocity vector $\vec{v}_n$ is

$$\vec{v}_n \equiv -\frac{\partial I}{\partial t} \left( \frac{\partial I}{\partial x} \right) \left( \frac{\partial I}{\partial y} \right) \cdot$$

Hint: Substitute something appropriate into $(v_x, v_y)$ in the motion constraint equation.

2. Consider a checkerboard pattern, oriented in the usual way namely with horizontal and vertical edges. Suppose the checkerboard is moving with image velocity $(v_x, 0)$.

What are the two motion constraint lines available to the vision system from this moving pattern?

3. Consider a video that is a sine function

$$I(x, y, t) = I_0 + \sin\left(\frac{2\pi}{N} k_0 x + \frac{2\pi}{T} \omega_0 t\right).$$

For fixed $t$, this is a sine function over $x$. For fixed $x$, this is a sine function of time, namely it oscillates over time $t$ with frequency $\omega$ cycles per $T$ time steps.

To make this sinusoidal image translate with 2D velocity $(v_x, 0)$, we must find suitable values for $k_0$ and $\omega_0$. Use the motion constraint equation to show that these values satisfy:

$$v_x = -\frac{\omega_0}{k_0} \frac{N}{T}.$$

Does this relationship makes sense in terms of units? (cycles per image, pixels per images, etc)

4. Same question, but now let the frequency be $(k_0, k_1, \omega_0)$ so

$$I(x, y, t) = I_0 + \sin\left(\frac{2\pi}{N} (k_0 x + k_1 y) + \frac{2\pi}{T} \omega_0 t\right).$$

What is the relationship between the image velocity vector and the frequency $(k_0, k_1, \omega_0)$? What is the normal velocity vector $\vec{v}_n$?

5. Suppose a set of vertically oriented lines is travelling to the right at speed $v_x$ and a set of horizontally oriented lines is travelling upwards at speed $v_y$.

What is the velocity of the points of intersection of these lines?
6. Consider a bright disk on a dark background. Suppose the disk is moving with velocity \( (1, 0) \). What are the normal velocities of points on the boundary of the disk? Answer this question for specific \((\cos \theta, \sin \theta)\) on the disk boundary, namely \(\theta = 0, 45, 90, 135, 180\) degrees.

7. For any image velocity \((v_x, v_y)\), the set of normal velocity vectors \(\vec{v}_n\) that are each consistent with this image velocity lie a circle in velocity space \((v_x, v_y)\), namely a circle containing the true velocity vector and the origin \((0,0)\). Use a simple geometric argument to show why this is so.

   Hint: Consider the set of all motion constraint lines that pass through the true velocity vector.

8. For the previous question, consider the normal velocity vectors that have length near 0, which are the vectors near O on the circle. Describe what spatial pattern gives rise to these normal velocities. What are the space-time Gabors that best detect these spatial patterns under the given motion?
Solutions

1. The normal velocity vector $\vec{v}_n$ is parallel to the image gradient direction $(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y})$ so we want to find the value $c$ such that

$$\vec{v}_n = c \left( \frac{\partial I}{\partial x}, \frac{\partial I}{\partial y} \right).$$

The normal velocity satisfies the motion constraint equation, so we substitute $v_n$ into

$$\frac{\partial I}{\partial x} v_x + \frac{\partial I}{\partial y} v_y + \frac{\partial I}{\partial t} = 0$$

which gives

$$c \left( (\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2 \right) + \frac{\partial I}{\partial t} = 0.$$

So

$$c = -\frac{\frac{\partial I}{\partial t}}{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2}.$$

2. For the horizontal lines of the checkerboard, $\frac{\partial I}{\partial y}$. Since there is no motion component in the $y$ direction, $\frac{\partial I}{\partial t} = 0$ across any horizontal edge. Substituting into the motion constraint equation gives the motion constraint line

$$v_y = 0$$

which corresponds to the $v_x$ axis.

For the vertical lines of the checkerboard, $\frac{\partial I}{\partial y} = 0$, and so the motion constraint line is $\frac{\partial I}{\partial x} v_x + \frac{\partial I}{\partial t} = 0$ or

$$v_x = -\frac{\partial I}{\partial t} / \frac{\partial I}{\partial x}$$

which is parallel to the $v_y$ axis. By definition, these two lines intersect at the true velocity $(-\frac{\partial I}{\partial t} / \frac{\partial I}{\partial x}, 0)$.

3.

$$\frac{\partial I(x(t), t)}{\partial x} = \frac{2\pi}{N} k_0 \cos \left( \frac{2\pi}{N} k_0 x + \frac{2\pi}{T} \omega t \right)$$

$$\frac{\partial I(x(t), t)}{\partial y} = 0$$

$$\frac{\partial I(x(t), t)}{\partial t} = \frac{2\pi}{T} \omega \cos \left( \frac{2\pi}{N} k_0 x + \frac{2\pi}{T} \omega t \right)$$

Substituting into the motion constraint equation gives

$$\left( \frac{2\pi}{N} k_0 v_x + \frac{2\pi}{T} \omega \right) \cos \left( \frac{2\pi}{N} k_0 x + \frac{2\pi}{T} \omega t \right) = 0$$
which implies the term on the left is 0. Isolating \( v_x \) gives the answer.

\[
v_x = -\frac{\omega_0}{k_0} \frac{N}{T}.
\]

What about units? \( \omega_0 \) is cycles per \( T \) frames and so \( \omega_0/T \) is cycles per frame. \( k_0 \) is cycles per \( N \) pixels, so \( k_0/N \) is cycles per pixel. So, \( \frac{\omega_0}{k_0} \frac{N}{T} \) is pixels per frame, which makes sense for an image velocity \( v_x \).

4. The \( x \) and \( t \) derivatives of \( I(x, y, t) \) are the same as in the previous question, but now the \( y \) derivative is

\[
\frac{\partial I(x(t), t)}{\partial y} = \frac{2\pi}{N} k_1 \cos(\frac{2\pi}{N} (k_0 x + k_1 y) + \frac{2\pi}{T} \omega t).
\]

Substituting into the motion constraint equation gives

\[
\frac{2\pi}{N} (k_0 v_x + k_1 v_y) + \frac{2\pi}{T} \omega = 0.
\]

This is just the motion constraint equation. (Note that the coefficients of this equation are proportional but not identical to the partial derivatives of the image intensity, since the latter include a cosine factor.)

To get the normal velocity vector, substitute as in Q1 into

\[
\vec{v}_n \equiv \frac{-\frac{\partial I}{\partial y}}{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2} \left( \frac{\partial I}{\partial x}, \frac{\partial I}{\partial y} \right).
\]

The cosine terms will all cancel, and we are left with:

\[
v_n = -\frac{N}{T} \frac{\omega_0}{k_0^2 + k_1^2} (k_0, k_1).
\]

which is a more general version of the one in the previous question.

5. The velocity is \((v_x, v_y)\). There is nothing to compute here.

6. The normal velocity in general is the true velocity minus the component of the true velocity that is perpendicular to the spatial gradient \((\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y})\). To compute the normal velocity, you can subtract off the component of the true velocity that is perpendicular to the spatial gradient. In this question, the true velocity is \((1, 0)\).

For \( \theta = 0 \) and 180 deg, the spatial gradient is perpendicular to \((0, 1)\). The true velocity has no component in that direction, and so the normal velocity equals the true velocity (for both cases).

For \( \theta = 90 \), the spatial gradient is perpendicular to \((1, 0)\) which is the true velocity, and so the normal velocity is \((0, 0)\). (Same for \( \theta = -90 \).)

For \( \theta = 45 \), the spatial gradient is perpendicular \((1, -1)\). We subtract from the true velocity the component in direction \((1, -1)\). This gives the normal velocity \((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\). Note the normal velocity vector has length \(\frac{1}{\sqrt{2}}\).
It is easy to make a calculation error for the last one, so let’s think about it differently. The motion constraint line must contain the true velocity \((1, 0)\) and it also must have a slope in direction \((1, -1)\), namely perpendicular to the spatial gradient. Thus by inspection the motion constraint line must also pass through \((0, 1)\). The normal velocity is the closest point on this line to the origin which, by inspection, is \((\frac{1}{2}, \frac{1}{2})\).

Similar reasoning holds for the disk point at 135 deg. The motion constraint line again must pass through \((1, 0)\) but now it must have slope in direction \((1, 1)\). Thus the motion constraint line must pass through \((0, -1)\). The closest point on this line to the origin is \((\frac{1}{2}, -\frac{1}{2})\). This is the normal velocity.

7. Recall from basic geometry that if you have two points \(O\) and \(V\) – say the origin and the true velocity vector in this case – then these two points define a unique circle such that \(OV\) is the diameter (which is obvious). See the figure below. The circle is dashed.

Next consider the set of all motion constraint lines that pass through \((v_x, v_y)\) in velocity space. We are trying to show that where each of these lines intersects the circle, we get a right angle between the line segment from the origin to the intersection point (the normal velocity) and the motion constraint line. To see this, recall a basic property of a circle defined by a diameter \(OV\), namely that for all points \(P\) on the circle, the angle \(OPV\) is a 90 degrees. But this is exactly the property that the normal velocity vectors have. Thus, the normal velocities indeed lie on that circle.

8. These normal velocities come from spatial pattern whose gradient is perpendicular to the true motion direction. The length of the normal velocity vector is (near) zero because the such spatial patterns do not change over time with that motion.

The space-time Gabors that best detect this motion have a spatial orientation in the direction of the motion and don’t have any time dependence, i.e. \(\omega = 0\).