Questions

1. Recall from the lecture that
   \[ F \ B(x) = \frac{1}{2}(1 + \cos(\frac{2\pi}{N}k)). \]
   What is the Fourier transform of \( B(x) * B(x) \) ?

2. (a) What is the Fourier transform of the shifted impulse function?
   \[ s(x) = \delta(x - x_0) = \begin{cases} 1, & x = x_0 \\ 0, & \text{otherwise} \end{cases} \]
   (b) What is the Fourier transform of \( I(x) * s(x) \) ?

3. What is the Fourier transform of
   \[ f(x) = \delta(x - 1) + \delta(x) + \delta(x + 1) \]?
   This is similar to the function \( B(x) \), but the weights on the neighbors are now different.
   How would you characterize this function? Lowpass? Bandpass? High pass? (defined in next lecture)

4. Recall the \( B(x) \) blur function from the lectures. In particular, note that \( \hat{B}(k) = 0 \) when \( k = \frac{N}{2} \). Why? Give an intuitive explanation.

5. Consider the following filter:
   \[ f(x) = \begin{cases} -2, & x = 0 \\ 1, & x = 1, -1 \\ 0, & \text{otherwise} \end{cases} \]
   (a) What is the amplitude spectrum of \( f(x) \) ? (i.e. amplitude as a function of frequency)
   (b) What is the phase spectrum of \( f(x) \) ?
   (c) How would you characterize this function? Lowpass? Bandpass? High pass?
   (d) ASIDE (in case you are curious). The filter can be used to compute an approximation of a second derivative. Why? That is, why does it make sense to use those particular coefficients for the second derivative?
      Hint: how would estimate the first derivative at a point halfway between two samples, e.g. \( x_0 \pm \frac{1}{2} \)? The second derivative is just the difference of these two first derivatives.

6. Show that if \( I(x) \) is real for all \( x \) then
   \[ \overline{I(k)} = \hat{I}(N - k). \]
   Of course, \( I(x) \) is real for all \( x \) when we are talking about images. However, there will be examples where we’ll take the Fourier transform of a function that has both real and imaginary parts. See Example 2 above.
   Note: This is the Conjugacy Property in the linear system notes (p 10).
Solutions

1. Use the convolution theorem.

\[ F \hat{B}(k) = (\frac{1}{2}(1 + \cos(\frac{2\pi}{N} k)))^2 \]

2. (a) \[ \hat{s}(k) = e^{-i \frac{2\pi}{N} k x_0} \]

(b) From the convolution theorem,

\[ F (I(x) * s(x)) = \hat{I}(k)\hat{s}(k) = \hat{I}(k)e^{-i \frac{2\pi}{N} k x_0} \]

Thus, each sine and cosine component of \( I(x) \) undergoes a phase shift of \( \frac{2\pi}{N} k x_0 \) radians. Note that phase shifts are modulo \( 2\pi \) radians.

3.

\[ \hat{f}(k) = \sum_{x=0}^{N} f(x)e^{-i \frac{2\pi}{N} k x} \]

\[ = e^{-i \frac{2\pi}{N} k(N-1)} + e^{-i \frac{2\pi}{N} k0} + e^{-i \frac{2\pi}{N} k1} \]

\[ = e^{i \frac{2\pi}{N} k} + 1 + e^{-i \frac{2\pi}{N} k} \]

\[ = 1 + 2 \cos(\frac{2\pi}{N} k) \]

You might have been tempted to say it was lowpass since it seems to be blurring. However, even though the amplitude spectrum is largest at \( k = 0 \) and falls off as \( k \) increases, The amplitude spectrum reaches 0 before \( k = \frac{N}{2} \) and in fact rises again and is non-zero at \( k = \frac{N}{2} \). So, strictly speaking, it is not low-pass. (Nor is it bandpass or highpass.)

4. Consider a cosine or sine function with frequency of \( k = \frac{N}{2} \). Such a function is of the form \( (\ldots, -1, 1, -1, 1, -1, \ldots) \), i.e. \( k = \frac{N}{2} \) corresponds to a wavelength (or “period”) of two pixels. By inspection, convolving such a function with \( B(x) \) produces 0 everywhere.

5. (a)

\[ \hat{f}(k) = \sum_{x=0}^{N-1} f(x)e^{-i \frac{2\pi}{N} k x} \]

\[ = e^{-i \frac{2\pi}{N} k(N-1)} - 2e^{-i \frac{2\pi}{N} k0} + e^{-i \frac{2\pi}{N} k1} \]

\[ = 2 \cos(\frac{2\pi}{N} k) - 2 \]

which is always negative. So, taking the magnitude gives

\[ |\hat{f}(k)| = 2 - 2 \cos(\frac{2\pi}{N} k) \]

(b) It is non-ideal high pass. It has value 0 when \( k = 0 \), and rises to its maximum value at \( k = \frac{N}{2} \).
(c) The phase spectrum of \( f(x) \) is \( \pi \) for all \( k \), i.e. \( \hat{f}(k) \) has no imaginary component, and its real component is negative for all \( k \).

(d) The second derivative is the derivative of the (first) derivative. If we have a function \( I(x) \), we can estimate what the derivative would be at \( x + \frac{1}{2} \) by

\[
\frac{I(x + 1) - I(x)}{(x + 1) - x} = I(x + 1) - I(x)
\]

and similarly we can estimate what the derivative would be at \( x - \frac{1}{2} \) by \( I(x) - I(x - 1) \). The second derivative at \( x \) could then be estimated by the difference of these two first derivatives,

\[
\frac{I(x + 1) - I(x) - (I(x) - I(x - 1))}{(x + \frac{1}{2}) - (x - \frac{1}{2})} = I(x + 1) - 2I(x) + I(x - 1).
\]

6. This involves some complex number trickery, but its good to practice.

\[
\hat{I}(N - k) = \sum_{x=0}^{N-1} I(x) e^{-i \frac{2\pi}{N} (N-k)x}
\]

\[
= \sum_{x=0}^{N-1} I(x) e^{-i \frac{2\pi}{N} x} e^{i \frac{2\pi}{N} kx}
\]

\[
= \sum_{x=0}^{N-1} I(x) e^{i \frac{2\pi}{N} kx}, \text{ since } e^{i2\pi x} = 1 \text{ for any integer } x
\]

\[
= \sum_{x=0}^{N-1} I(x) e^{i \frac{2\pi}{N} kx}, \text{ i.e. the conjugate of the conjugate}
\]

\[
= \sum_{x=0}^{N-1} I(x) e^{-i \frac{2\pi}{N} kx} \text{ i.e. since } I(x) \text{ is real}
\]

\[
= \hat{I}(k)
\]