Questions

1. Suppose we have an image \( I(x, y) \) and a function \( f(x, y) \). If we wish to interpret \( f(x, y) \) as the receptive field profile of a family of “cells”, then the cross-correlation

\[
f \otimes I(x, y) \equiv \sum_{x'} \sum_{y'} f(x' - x, y' - y) I(x', y')
\]

is naturally interpreted as the responses of such cells, namely, at each position \( (x, y) \) there is one cell with this receptive field profile, and one computes the inner product of this receptive field profile with the image.

Recall the definition of 2D convolution:

\[
h \ast I(x, y) = \sum_{x'} \sum_{y'} h(x - x', y - y') I(x', y').
\]

and note it is different from cross-correlation.

(a) How could we choose \( h(\cdot) \) such that

\[
h \ast I(x, y) = f \otimes I(x, y) \ ?
\]

(b) Cross correlation \( f \otimes I \) can be visualized by taking a template \( f(\cdot) \), centering it at each position of an image \( I(\cdot) \), and taking the inner product. That is, cross-correlation can be thought of as the output of template \( f(\cdot) \) centered at different points in an image \( I(\cdot) \). How does this interpretation differ fundamentally from convolution \( h \ast I \) ?

2. Let \( D(x) \) and \( B(x) \) be defined as in lecture 14.

(a) What is \( D(x) \ast B(x) \) ?

(b) What is \( D(x) \ast D(x) \) ?

(c) What is \( B(x) \ast B(x) \) ?

3. Let \( f(t) \) be some function such that

\[
I(t) \ast f(t) = 2I(t) - 3I(t - 1) + I(t - 2).
\]

What is \( f(t) \) ?

4. Prove the associative law for 1-D convolution,

\[
(f \ast g) \ast h = f \ast (g \ast h)
\]

You may assume that each of the three functions are defined over all the integers, but only have non-zero values over a finite range.

5. What is the result of convolving a 1D sine function with \( h(x) \) ?

Hint: in the lecture, I showed the result of convolving a 1D cosine function with \( h(x) \). Use a similar idea here.
Solutions

1. With convolution $h * I$ as defined in the question, the function $h$ is not a template, but rather it is an impulse response. It says how the output function $h * I$ depends on each value of the input function $I$, namely each input value of $I(x', y')$ is the weight of an impulse, and the response of that impulse is $I(x', y')h(x - x', y - y')$ which is a function of $(x, y)$.

   The key concept to distinguish here is that cross-correlation expresses how each output value is determined by the image and the template, whereas convolution expresses how each input value contributes to the output image.

2. (a) 
   
   
   
   \[
   D(x) * B(x) = \begin{cases} 
   \frac{1}{8}, & x = -2 \\
   \frac{1}{4}, & x = -1, \\
   -\frac{1}{4}, & x = 1, \\
   -\frac{1}{8}, & x = 2 \\
   0, & \text{otherwise}
   \end{cases}
   \]

   Verify in Matlab by running `conv([.5 0 -.5], [.25, .5, .25])`

   (b) 
   
   
   
   \[
   D(x) * D(x) = \begin{cases} 
   \frac{1}{4}, & x = 2, -2 \\
   -\frac{1}{2}, & x = 0, \\
   0, & \text{otherwise}
   \end{cases}
   \]

   To verify, in Matlab run `conv([.5 0 -.5], [.5 0 -.5])`

   (c) 
   
   
   
   \[
   B(x) * B(x) = \begin{cases} 
   \frac{1}{16}, & x = 2, -2 \\
   \frac{1}{4}, & x = -1, 1 \\
   \frac{3}{8}, & x = 0 \\
   0, & \text{otherwise}
   \end{cases}
   \]

   To verify, in Matlab run `conv([.25 .5 .25], [.25 .5 .25])`. Also, notice that the result sums to 1, just as $B(x)$ sums to 1.

3. $f(0) = 2, f(1) = -3, f(2) = 1$. Or 

   \[
   f(t) = 2\delta(t) - 3\delta(t - 1) + \delta(t - 2)
   \]
4. All summations in the following are over $\infty, \ldots, \infty$.

$$(f * g)(x) * h(x) = \sum_{x'} (f * g)(x') \ h(x - x')$$

$$= \sum_{x'} \sum_{x''} f(x'') g(x' - x'') \ h(x - x')$$

$$= \sum_{x''} f(x'') \sum_{x'} g(x' - x'') \ h(x - x')$$

let $x' - x'' = v$

$$= \sum_{x''} f(x'') \sum_{v} g(v) \ h((x - x'') - v)$$

$$= \sum_{x''} f(x'')(g * h)(x - x'')$$

$$= (f * (g * h))(x)$$

5.

$h(x) \sin\left(\frac{2\pi}{N} k x\right) = \sum_{x'=0}^{N-1} h(x') \sin\left(\frac{2\pi}{N} (x - x')\right)$

Recalling the trigonometry identity

$$\sin(\alpha + \beta) = \sin \alpha \sin \beta + \cos \alpha \cos \beta$$

we can expand the $\sin()$ on the right side of the summation, and so the right hand side is just a sum of sine and cosine functions with variable $x$ and constant frequency $k$, namely

$$h(x) \sin\left(\frac{2\pi}{N} k x\right) = a \cos\left(\frac{2\pi}{N} k x\right) + b \sin\left(\frac{2\pi}{N} k x\right)$$

where

$$a = \sum_{x'=0}^{N-1} h(x') \cos\left(\frac{2\pi}{N} x'\right)$$

$$b = \sum_{x'=0}^{N-1} h(x') \sin\left(\frac{2\pi}{N} x'\right)$$

which are just the inner products of the N dimensional vectors $h(\cdot)$ with a cosine or sine of frequency $k$, respectively.