Questions

1. (a) Suppose the ground plane surface is slanted at some slope \( m > 0 \) relative to the observer’s \( Z \) axis, so 
\[
Z = Z_0 + mY .
\]
Give an expression for the \( \Delta x \) and \( \Delta y \) that results from steps on the surface \((1, 0, 0)\) and \((0, \cos \sigma, \sin \sigma)\), where \( \sigma \) is the slant, i.e. \( m = \tan \sigma \). You should make a small angle approximation for \((x, y)\) but not for \( \sigma \).

(b) Write the ratio \( \frac{\Delta y}{\Delta x} \) in terms of slant \( \sigma \) and any other quantities that it depends on.

2. When the user’s \( Z \) axis is parallel to the ground, the horizon is at \( y = 0 \). Where is the horizon for the plane in the previous question?

3. Many animals such as birds, snakes, fish and deer have white bellies. It has been claimed that such surface pigmentation decreases the visibility of the animal (i.e. “camouflage”) when it is viewed from the side under natural lighting. Briefly justify this claim.

4. Consider the linear shading model which assumes a low relief surface. Suppose we have a depth function 
\[
Z(X,Y) = Z_0 + a \sin(k_0 Y)
\]
and a light source direction from above \((0, L_y, -L_z)\). In this case, the linear shading model 
\[
I(X,Y) = (\frac{\partial Z}{\partial X}, \frac{\partial Z}{\partial Y}, -1) \cdot (L_X, L_Y, L_Z)
\]
reduces to:
\[
I(X,Y) = L_y \frac{\partial Z(X,Y)}{\partial Y} - L_z.
\]

(a) How big should \( a \) be so that we can ensure a linear model? (The linear model assumes a low relief surface, namely small gradients.)

(b) Describe the resulting image in terms of the sinusoidal variation of the depth. Where does the intensity peak occur? What is the frequency of the intensity variations?

(c) What if the depth function were a sinusoid that varied with \( X \) rather than \( Y \), and the same lighting were present? What would be the shading image in this case?
Solutions

1. Dividing $Z = Z_0 + mY$ by $Z$ gives

$$1 = m \frac{Y}{Z} + \frac{Z_0}{Z}.$$

Then, since $y = \frac{Y}{Z}$, where $y$ is measured in radians (small angle approximation), we have

$$1 = my + \frac{Z_0}{Z}.$$

from which we can write $y$ in terms of $\frac{1}{Z}$.

Taking a small step size in the image $\Delta y$ and approximating $\frac{\Delta y}{\Delta Z} \approx \frac{dy}{dZ}$ gives

$$\Delta y \approx Z_0 \frac{\Delta Z}{mZ^2}.$$

So when $\Delta Z = \sin \sigma$ we have

$$\Delta y \approx \frac{Z_0 \sin \sigma}{\tan \sigma Z^2} = \frac{Z_0 \cos \sigma}{Z \cdot Z}.$$

What about steps in the $x$ direction? A step $\Delta X = 1$ in 3D does not lead to a change in $Z$ for this surface. So, since $x = \frac{X}{Z}$ in radians, we have $\Delta x = \frac{1}{Z}$. This is the same is for the ground plane discussed in class.

The compression ratio for the obliquely slanted plane is

$$\frac{\Delta y}{\Delta x} = \frac{Z_0 \cos \sigma}{Z}.$$

The $Z_0$ is there to set the scale of the scene. From a single view and using a pinhole camera model (i.e. no lens), there is no way to know if the scene is a little toy doll’s house room or whether it is the size of Canada. So if $Z > Z_0$, i.e. $Y > 0$, then we have compression beyond the basic effect of the surface slant, and if $Z < Z_0$ i.e. $Y < 0$, then we have expansion (not obvious, but that’s what the math says).

2. From

$$1 = my + \frac{Z_0}{Z}$$

we let $Z \to \infty$. This gives us the new horizon, namely $y = \frac{1}{m}$.

3. If an animal has a uniform color, then under natural lighting the top part of the animal will have the highest luminance and the bottom part will have the lowest luminance. This is because light tends to be from above (sunlight, cloudy day). The white colored belly partially cancels out this shadow/shading effect, since the illumination and reflectance gradients go in opposite directions. This phenomena is called *countershading*. It is a well known principle of camouflage.
4. (a) We want the surface gradient to be small, i.e. low relief. So, taking the derivative of \( Z \) with respect to \( Y \) gives

\[
\frac{\partial Z}{\partial Y} = a \ k_0 \ \cos(k_0Y)
\]

and we want that to be small. So we want \( a \) to be much less than \( \frac{1}{k_0} \). That is, the higher the spatial frequency, the smaller we require that the amplitude is, in order to ensure the depth gradient is small.

(b) Plugging \( \frac{\partial Z}{\partial Y} \) from (a) into the linear shading model which was stated in the question now gives

\[
I(X,Y) = L_y \ a \ k_0 \ \cos(k_0Y) - L_z.
\]

That is, the intensity will be 90 degrees out of phase with the depth map. So the peak in intensity will occur in the middle of the surface slope, not at depth max or min. The intensity function will have the same frequency \( k_0 \) as the depth map.

(c) Applying the linear shading model with respect to the vertical depth sinusoid

\[
Z(X,Y) = Z_0 + a \ \sin(k_0X)
\]

gives

\[
I(X,Y) = \left( \frac{\partial Z}{\partial X}, \frac{\partial Z}{\partial Y}, -1 \right) \cdot (L_X, L_Y, L_Z) = \left( \frac{\partial Z}{\partial X}, 0, -1 \right) \cdot (L_X, 0, L_Z)
\]

or

\[
I(X,Y) = -L_z
\]

that is, there would be no shading, i.e. The surface would have constant intensity, \( -L_z > 0 \). This may seem counterintuitive, but in fact it is easy to demonstrate. Email/see me if you are unsure.