COMP 546

Lecture 16

Linear Systems 3:
white noise, filtering

Tues. March 14, 2017
Fourier transform

\[ \hat{I}(k) \equiv F \hat{I}(x) = \sum_{x=0}^{N-1} I(x) e^{-i \frac{2\pi}{N} kx} \]

\[ F_{k,x} = e^{-i \frac{2\pi}{N} kx} \]
\[ F I(x) = |\hat{I}(k)| e^{-i \phi(k)} \]
Periodicity Property

For any integer $m$,

$$cos\left(\frac{2\pi}{N} kx\right) = cos\left(\frac{2\pi}{N} (k + mN) x\right)$$

$$sin\left(\frac{2\pi}{N} kx\right) = sin\left(\frac{2\pi}{N} (k + mN) x\right)$$

It follows immediately that $\hat{I}(k) = \hat{I}(k + mN)$.

This is called the periodicity property of the Fourier transform.
Conjugacy Property

For any integer $m$,

\[
\cos\left(\frac{2\pi}{N} kx\right) = \cos\left(\frac{2\pi}{N} (N - k) x\right)
\]

\[
\sin\left(\frac{2\pi}{N} kx\right) = -\sin\left(\frac{2\pi}{N} (N - k) x\right)
\]

These suggest that $\hat{I}(k) = \hat{I}(N - k)$.

In fact, this property only holds if $I(x)$ is real.
Noise

Let $n(x)$ be independent noise e.g. Gaussian with mean 0. What can we say about the Fourier transform of $n(x)$?

Claim: on average i.e. expected value

$$E \left\{ |F n(x)|^2 \right\} = E \left\{ |\hat{n}(k)|^2 \right\} = \text{constant}$$
“White” noise

white noise signal with mean 0, $\sigma_n = 1.036$

$n(x)$

$|\hat{n}(k)|^2$

power spectrum (up to Nyquist frequency)

one trial (example above)
average of 50 trials
White noise: it doesn’t matter if noise is Gaussian or binary.
Convolution Theorem

\[ F \{ I(x) \ast (x) \} = F(I(x)) \cdot F(x) \]

\[ = \hat{I}(k) \hat{h}(k) \]

\[ = |\hat{I}(k)| \cdot |\hat{h}(k)| \cdot e^{-i \phi_I(k)} \cdot e^{-i \phi_h(k)} \]

Convolving an image \( I(x) \) with a filter \( (x) \) changes the amplitude and phase of each frequency component.
Ideal Low Pass Filter

\[ |\hat{h}(k)| = \begin{cases} 
1, & \text{if } 0 < k \leq k_0 \\
0, & \text{if } k_0 < k < \frac{N}{2}
\end{cases} \]
Ideal High Pass Filter

\[ |H(k)| = \begin{cases} 1, & \text{if } \ell_1 \leq k < \frac{n}{2} \\ 0, & \text{if } 0 < k < \ell_1 \end{cases} \]
Ideal Band Pass Filter

\[ |\hat{h}(k)| = \begin{cases} 1, & \text{if } 0 < k_1 < k < k_2 < \frac{N}{2} \\ 0, & \text{otherwise} \end{cases} \]
bandwidth

\[ k_2 - k_1 \]

octave bandwidth

\[ \equiv \log_2 k_2 - \log_2 k_1 \]

\[ = \log_2 \left( \frac{k_2}{k_1} \right) \]

e.g. one octave means \[ \frac{k_2}{k_1} = 2 \]
Non-Ideal Filters

low pass

band pass

high pass
Bandwidth of Non-Ideal Bandpass Filter

[Graph showing a function with maxima at $k_1$ and $k_2$, and bandwidth "at half height"]
Example 1: Non-ideal low pass

\[ B(x) = \begin{cases} \ldots & \text{for } x = -2, -1, 0, 1, 2 \end{cases} \]

\[ \hat{B}(k) = \frac{1}{2} \left( 1 + \cos \left( \frac{2\pi k}{N} \right) \right) \]

Diagram: A plot of \( \hat{B}(k) \) with peaks at \( k = -\frac{N}{2}, 0, \frac{N}{2} \).
\[ \mathbf{B}(x) = \begin{array}{c}
-2 \\
-1 \\
0 \\
1 \\
2 \\
\end{array} \]

\[ \mathbf{B}(x) \ast \mathbf{B}(x) = \begin{array}{c}
-2 \\
-1 \\
0 \\
1 \\
2 \\
\end{array} \]

\[ \left| \mathbf{\hat{B}}(k) \right|^2 = \left| \frac{1}{2} \left( 1 + \cos \left( \frac{2\pi}{N} k \right) \right) \right|^2 \]
B(x) * ... * B(x), i.e. \( m \) times

\[ B(k)^m = (0.5 + 0.5\cos(2\pi N k))^m \]
Example 2: Non-ideal low pass (Gaussian)

\[ G(x, \sigma) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{x^2}{2\sigma^2}} \]

\[ F G(x, \sigma) \approx e^{-\left(\frac{2\pi}{N}\right) \frac{k^2\sigma^2}{2}} \]

with equality in the limit as distance between samples goes to 0 (and N goes to infinity), i.e. continuous Fourier transform.
Example 2:  Non-ideal low pass (Gaussian)

\[ G(x, \sigma) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{x^2}{2\sigma^2}} \]

\[ F[G(x, \sigma)] \approx e^{-\left(\frac{2\pi}{N}\right)\frac{k^2\sigma^2}{2}} \]

with equality in the limit as distance between samples goes to 0 (and N goes to infinity), i.e. continuous Fourier transform.
Example: Non-ideal band pass (DOG)
Convolution Theorem (version 2)

\[ F \{ I(x) \ast x \} = \frac{1}{N} F I(x) F(x) \]

Proof (omitted):

Take the right side below and crank through to get the left side.

\[ I(x) \ast x = \frac{1}{N} F^{-1} \{ F I(x) F(x) \} \]
Fourier transform of a cosine Gabor

\[
\mathcal{F} \ G_c(x, \sigma, k_0) = \frac{1}{N} \mathcal{F} \ G(x, \sigma) \star \mathcal{F} \ \cos(\frac{2\pi k_0}{N} x)
\]

\[
= \frac{1}{N} e^{-\frac{1}{2} (\frac{2\pi}{N} k_0 \sigma)^2} \star \frac{N}{2} \left( \delta(k - k_0) + \delta(k + k_0) \right)
\]
Fourier transform of a cosine Gabor

\[
\begin{align*}
\hat{F}\ G_c(x,\sigma, k_0) &= \frac{1}{N} \hat{F}\ G(x,\sigma) * \hat{F}\ \cos\left(\frac{2\pi}{N} k_0 x\right) \\
&= \frac{1}{N} e^{-\frac{1}{2} \left(\frac{2\pi}{N} k_0 \sigma\right)^2} * \frac{N}{2} (\delta(k-k_0) + \delta(k+k_0)) \\
&= \frac{1}{2} (\hat{G}(k-k_0, \sigma) + \hat{G}(k+k_0, \sigma))
\end{align*}
\]

Band pass!

See Exercises for sine Gabor.
2D Fourier Transform

\[ \hat{I}(k_x, k_y) = \sum_{y=0}^{N-1} \sum_{x=0}^{N-1} I(x, y) e^{-i \left( \frac{2\pi}{N} (k_x x + k_y y) \right)} \]

\[ = \sum_{y=0}^{N-1} e^{-i \frac{2\pi}{N} k_y y} \sum_{x=0}^{N-1} I(x, y) e^{-i \frac{2\pi}{N} k_x x} \]

\[ = \left| \hat{I}(k_x, k_y) \right| e^{i \phi(k_x, k_y)} \]

phase spectrum

amplitude spectrum
Spatial domain $I(x, y)$  

Frequency domain $\hat{I}(k_x, k_y)$
Properties of 2D Fourier Transform

\[
F \{ I(x, y) \} = \hat{I}(k_x, k_y) = |\hat{I}(k_x, k_y)| e^{-i \phi_I(k_x, k_y)}
\]

\[
F \{ I(x, y) \} = F I(x, y) F (x, y)
\]

\[
F \{ I(x, y) \} = \frac{1}{N} F I(x, y) F (x, y)
\]

\[
\hat{I}(k_x, k_y) = \hat{I}(k_x + m_x N, k_y + m_y N)
\]

\[
\hat{I}(k_x, k_y) = \hat{I}(N - k_x, N - k_y)
\]
Frequency domain: ideal bandpass filters $\hat{h}(k_x, k_y)$

Spatial domain

- Image [64 x 64]
- $|k|$ in [1, 2)
- $|k|$ in [2, 4)
- $|k|$ in [4, 8)
- $|k|$ in [8, 16)
- $|k|$ in [16, 32)
Spatial domain: ideal bandpass filtered images

\[ I(x, y) \]

(I didn’t show the filters \((x, y)\).)
2D Gaussian (non-ideal low pass)

Spatial domain

\[ G(x, y, \sigma) = \frac{1}{2\pi \sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \]

Frequency domain

\[ F G(x, \sigma) \approx e^{-\left(\frac{2\pi}{N}\right)\frac{(k_x^2+k_y^2)}{2}} \sigma^2 \]
Frequency domain non-ideal low pass filters
\( \hat{G}(k_x, k_y, \sigma) \)
Spatial domain: non-ideal low pass filtered images

\[ I(x, y) \cdot G(x, y, \sigma) \]
Frequency domain: non-ideal band pass filters

$$\hat{D}\hat{O}\hat{G}(k_x, k_y, \sigma, \_\_\_) = \hat{G}(k_x, k_y, \sigma) - \hat{G}(k_x, k_y, 1.1 \ \sigma)$$
Spatial domain: Non-ideal band pass filtered images

\[ I(x, y) = (G(x, y, \sigma) - G(x, y, 1.1 \sigma)) \]
Similar to lecture 5: What is $I(x, y)$ $\cos Gabor(x, y, \sigma)$?
Convolution with four cosine Gabors
Convolution with four sine Gabor filters
Fourier transform of 2D Gabor

See Exercises and Assignment 4
Fourier transform of 2D Gabor

See Exercises and Assignment 4