Last lecture I introduced the idea that any function could be written a sum of sinusoids. The reason we would like to do this is that sinusoids behave nicely under convolution, and convolution is an operation that vision systems and audition systems do, in some sense. For example, if we think of a family of cells that have linear responses and that all have the same spatio-temporal sensitivity, then we can model the responses of the family of all such cells as a convolution. Another example of convolution is that the eye’s optics, and the head’s effect on an incoming sound. If we would like to understand the signal (image/sound) transformation that occurs under convolution, then a useful way to do so is to write the function as a sum of sinusoids and understand what happens to each of the component sinusoids.

Note that the vision/auditory system doesn’t decompose signals into different frequency sinusoids. Rather, as we will see, it analyzes a signal by filtering it into bands or ranges of different frequencies in this way is called filtering. The concept of filtering by size (wavelength) is a familiar one. You can imagine having a large bag of rocks and wanting to sort the rocks into ranges of different sizes. You could first pass the rocks through a mesh that has big holes of a certain size, so only the big rocks would pass through. Then take the smaller rocks that didn’t pass through, and pass them through mesh filter that has smaller holes so that only the medium size rocks pass through. One would be left with three sets of rocks, each have a range of sizes.

A slightly different kind of filtering occurs in color vision. Here the L, M, and S receptors filter by frequency. Each cell is sensitive to light of a different range of wavelengths. There is some frequency overlap in the sensitivities – i.e. no hard cutoff. But the idea is the same. You have a signal that consists of frequencies of many wavelengths, and you decompose roughly into bands having different ranges of frequencies.

The above figure illustrates this idea of filtering. The top left panel shows a 1D signal, which we decompose as a sum of five ranges of frequency components which are shown in the other five panels. We will see a bit later today how this is done.
Here is another example, this time with a 2D image. Again we decompose the signal into a sum of five ranges of frequency components which are shown in the other five panels.

Let’s now turn to the mathematics of how this is done. The frequency decomposition is similar to what we saw last lecture, where sine and cosine functions were a basis of our space of signals. The main difference in the formulation is that we combine the sine and cosines of each frequency into a single basis vector and we do so using the real and imaginary part of a complex number.

**Discrete Fourier Transform**

Consider 1D signals $I(x)$ which are defined on $x \in \{0, 1, ..., N - 1\}$. Define the $N \times N$ Fourier transform matrix $F$ whose $k^{th}$ row and $x^{th}$ column is:

$$
F_{k,x} = \cos\left(\frac{2\pi}{N} k x\right) - i \sin\left(\frac{2\pi}{N} k x\right)
$$

$$
\equiv e^{-i\frac{2\pi}{N} k x}
$$

Note that this matrix is symmetric since $e^{-i\frac{2\pi}{N} k x} = e^{-i\frac{2\pi}{N} x k}$. Also note that each row and column of the matrix $F$ has a real part and an imaginary part. The real part is a sampled cosine function. The imaginary part is a sampled sine function. Note that the leftmost and rightmost column of the matrix ($x = 0$ and $x = N - 1$) are not identical. You would need to go to $x = N$ to reach the same value as at $x = 0$, but $x = N$ is not represented. Similarly, the first and last row ($k = 0$ and $k = N - 1$) are not identical.

Multiplying the Fourier transform matrix $F$ by the $N \times 1$ vector $I(x)$ defines:

$$
\hat{I}(k) \equiv F I(x) = \sum_{x=0}^{N-1} I(x) e^{-i\frac{2\pi}{N} k x}
$$

(1)

$\hat{I}(k)$ is called the discrete Fourier transform (DFT) of $I(x)$. In general, $\hat{I}(k)$ is a complex number for each $k$. We can write it using Euler’s equation:

$$
\hat{I}(k) = A(k) e^{i\phi(k)}
$$

$|\hat{I}(k)| = A(k)$ is called the amplitude spectrum and $\phi(k)$ is called the phase spectrum.
Periodicity property of the Fourier transform

The Fourier transform definition assumed that the function was defined on $x \in 0, \ldots, N - 1$, and for frequencies $k$ in $0, \ldots, N - 1$. However, sometimes we will want to be more flexible with our range of $x$ and $k$.

For example, we may want to consider functions $h(x)$ that are defined on negative values of $x$. Examples are the local difference function $D(x)$ and the local average function $B(x)$, the Gaussian function which has mean 0, Gabor functions, etc. Indeed sine and cosine functions are defined over all integers, so there is no obvious reason why the Fourier transform should be restricted to just $x \in 0$ to $N - 1$.

Similarly, $\hat{I}(k)$ is well-defined for any integer $k$. In particular, we are defining our frequencies as cycles per $N$ pixels, and $\hat{I}(k)$ is periodic in $k$ with period $N$,

$$\hat{I}(k) = \hat{I}(k + mN)$$

since, for any integer $m$,

$$e^{i \frac{2\pi}{N} kx} = e^{i \frac{2\pi}{N} (k + mN)x} = e^{i \frac{2\pi}{N} k} e^{i \frac{2\pi}{N} mN}$$

i.e.

$$e^{i \frac{2\pi}{N} mN} = \cos(2\pi m) + i \sin(2\pi m) = 1$$

Thus, if we use frequency $k + mN$ instead of $k$ in the definition of the Fourier transform, we get the same value. We will refer to this as the periodicity property.

Conjugacy property of the Fourier transform

It is a bit strange that our function $I(x)$ has $N$ points and we will write it in terms of $2N$ functions, namely $N$ cosines and $N$ sines. I mentioned this point last lecture as well, and showed that indeed only $N$ functions are needed, namely $\frac{N}{2} + 1$ cosines and $\frac{N}{2} - 1$ sines. This suggests that there is a redundancy in $\hat{I}(k)$ values, and indeed I gave the basic idea last lecture of what that redundancy is, namely that $\cos(\frac{2\pi}{N} kx) = \cos(\frac{2\pi}{N} (N - k)x)$ and so taking the inner product of $I(x)$ will give the same value. Similarly, $\sin(\frac{2\pi}{N} kx) = - \sin(\frac{2\pi}{N} (N - k)x)$ and so taking the inner product of $I(x)$ with these two functions will give the same number with opposite signs.

Conjugacy property: If $I(x)$ is a real valued function, then

$$\overline{\hat{I}(k)} = \hat{I}(N - k).$$

The property does not apply if $I(x)$ has imaginary components. See Example 4 below.
Proof:

\[
\hat{I}(N - k) = \sum_{x=0}^{N-1} I(x) e^{-i \frac{2\pi}{N} (N-k)x} \\
= \sum_{x=0}^{N-1} I(x) e^{-i 2\pi x} e^{i \frac{2\pi}{N} kx} \\
= \sum_{x=0}^{N-1} I(x) e^{i \frac{2\pi}{N} kx}, \text{ since } e^{i 2\pi x} = 1 \text{ for any integer } x \\
= \sum_{x=0}^{N-1} I(x) e^{-i \frac{2\pi}{N} kx} \\
= \sum_{x=0}^{N-1} \hat{I}(x) e^{-i \frac{2\pi}{N} kx}, \text{ if } I(x) \text{ is real} \\
= \sum_{x=0}^{N-1} \hat{I}(x) e^{-i \frac{2\pi}{N} kx} \\
= \hat{I}(k)
\]

Euler’s equation (again)

Some of our calculations of Fourier transforms below will use Euler’s equation, \(e^{i\theta} = \cos(\theta) + i \sin(\theta)\). You can verify for yourselves that:

\[
\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\
i \sin(\theta) = \frac{1}{2}(e^{i\theta} - e^{-i\theta})
\]

In particular, we will often take \(\theta = \frac{2\pi}{N} kx\).

Examples of Fourier transforms

Let’s look at some examples, starting with an impulse function, and the local difference and local average.

Example 1: Impulse function

Recall

\[
\delta(x) \equiv \begin{cases} 
1, & x = 0 \\
0, & \text{otherwise}
\end{cases}
\]
Its Fourier transform is

\[ \hat{\delta}(k) = \sum_{x=0}^{N-1} \delta(x) e^{-i \frac{2\pi}{N} kx} \]
\[ = 1 \cdot e^{i \frac{2\pi}{N} k \cdot 0} \]
\[ = 1 \]

This is rather surprising. It says that we can obtain an impulse function by summing a set of cosine functions over all frequencies \( k \in 0, 1, \ldots, N - 1 \). (Note that the phase is 0, i.e. \( \phi(k) = 0 \) for all \( k \), and so there are no sine terms.) Basically, what happens is that all the cosine functions have the value 1 at \( x = 0 \), whereas at other values of \( x \) there are a range of values, some positive and some negative, and these other values cancel each other out when you take the sum.

To try to illustrate what is going on here, I have written a Matlab script [http://www.cim.mcgill.ca/~langer/546/MATLAB/sumOfSinusoids.m](http://www.cim.mcgill.ca/~langer/546/MATLAB/sumOfSinusoids.m) which shows what happens when you add up all the cosines (top) and sines (bottom) of frequency \( k = 0, \ldots, N - 1 \) for some chosen \( N \).

**Example 2: local difference**

Recall the local difference function \( D(x) \) from last lecture. It has value \(-\frac{1}{2}\) at \( x = 1 \) and value \( \frac{1}{2} \) at \( x = -1 \).

\[ \hat{D}(k) = \sum_{x} D(x) e^{-i \frac{2\pi}{N} kx} \]
\[ = \frac{1}{2} (-1 \cdot e^{-i \frac{2\pi}{N} k} + 1 \cdot e^{-i \frac{2\pi}{N} (-1)}) \]
\[ = \frac{1}{2} (-e^{-i \frac{2\pi}{N} k} + e^{i \frac{2\pi}{N} k}) \]
\[ = i \sin\left(\frac{2\pi}{N} k\right) \]
\[ = e^{i \frac{\pi}{2}} \sin\left(\frac{2\pi}{N} k\right) \]

Notice that \( \hat{D}(k) \) is purely imaginary and the plot below shows the imaginary component only. The phase spectrum is constant \( \frac{\pi}{2} \).

**Example 3: local average**

\[ B(x) = \begin{cases} 
\frac{1}{2}, & x = 0 \\
\frac{1}{4}, & x = -1 \\
\frac{1}{4}, & x = 1 \\
0, & \text{otherwise}
\end{cases} \]

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Taking its Fourier transform,
\[
F B(x) = \frac{1}{2} + \frac{1}{4} (e^{-i\frac{2\pi}{N} k} + e^{-i\frac{2\pi}{N} k(-1)}) \\
= \frac{1}{2} + \frac{1}{4} (e^{-i\frac{2\pi}{N} k} + e^{i\frac{2\pi}{N} k}), \\
= \frac{1}{2} (1 + \cos(\frac{2\pi}{N} k))
\]

Notice that \( \hat{B}(k) \) is real, i.e. it has no imaginary component. Moreover it is non-negative. Thus, the phase spectrum \( \phi(k) \) is 0.

Example 4: the “complex exponential” \( e^{i\frac{2\pi}{N} k_0 x} \) for some integer frequency \( k_0 \)

Eventually below we will compute the Fourier transforms of \( \cos(\frac{2\pi}{N} k_0 x) \) and \( \sin(\frac{2\pi}{N} k_0 x) \) for some fixed integer \( k_0 \). To do so, we will write them as a sum and difference of functions \( e^{i\frac{2\pi}{N} k_0 x} \), respectively, and then use the following:
\[
F e^{i\frac{2\pi}{N} k_0 x} = N\delta(k - k_0).
\]

That is,
\[
\sum_{x=0}^{N-1} e^{i\frac{2\pi}{N} k_0 x} e^{-i\frac{2\pi}{N} k x} = \begin{cases} N, & k = k_0 \\ 0, & k \neq k_0 \end{cases}
\]

How to derive this? The case \( k = k_0 \) is easy since the exponent is just 0 and \( e^0 = 1 \) which we sum \( N \) times.

For the case \( k \neq k_0 \), we can use the following identity which you have all seen before. If \( \gamma \) be any number (real or complex) then
\[
(1 - \gamma) \sum_{x=0}^{N-1} \gamma^x = 1 - \gamma^N.
\]
and so
\[ \sum_{x=0}^{N-1} \gamma^x = 1 - \gamma^N \frac{1}{1 - \gamma}. \]

Applying this identity for \( \gamma = e^{i \frac{2\pi}{N}(k-k_0)} \) gives
\[ \sum_{x=0}^{N-1} e^{i \frac{2\pi}{N}(k-k_0)x} = \frac{1 - e^{i 2\pi(k-k_0)}}{1 - e^{i \frac{2\pi}{N}(k-k_0)}}. \]

The numerator on the right hand side vanishes because \( k - k_0 \) is an integer and so
\[ e^{i 2\pi(k-k_0)} = 1. \]

What about the denominator? Since \( k \) and \( k_0 \) are both in \( 0, \ldots, N-1 \) and since we are considering the case that \( k \neq k_0 \), we know that \( |k-k_0| < N \) and so \( e^{i \frac{2\pi}{N}(k-k_0)} \neq 1 \). Hence the denominator does not vanish. Since the numerator vanishes but the denominator does, we can conclude
\[ \frac{1 - e^{i 2\pi(k-k_0)x}}{1 - e^{i \frac{2\pi}{N}(k-k_0)}} = 0. \]

This completes the derivation for the case \( k \neq k_0 \).

**Example 5: constant function**

In the last example, call it \( h(x) = e^{i \frac{2\pi}{N}k_0x} \), if we take \( k_0 = 0 \) then we just have a constant function, namely
\[ h(x) = 1. \]

In this case,
\[ \hat{h}(k) = N \delta(k). \]

Thus, the Fourier transform of the constant function \( h(x) = 1 \) is a delta function \textit{in the frequency domain}, namely it has value \( N \) at \( k = 0 \) and has value 0 for all values of \( k \) in \( 1, \ldots, N-1 \).

**Examples 6 and 7: cosine and sine**

We use Euler’s equation to write cosine and sine in terms of complex exponentials.
\[
\mathbf{F} \cos\left(\frac{2\pi}{N}k_0x\right) = \sum_{x=0}^{N-1} \cos\left(\frac{2\pi}{N}k_0x\right)e^{-i \left(\frac{2\pi}{N}kx\right)}
= \sum_{x=0}^{N-1} \frac{1}{2} \left(e^{i \frac{2\pi}{N}k_0x} + e^{-i \frac{2\pi}{N}k_0x}\right)e^{-i \frac{2\pi}{N}kx}
= \frac{N}{2} \left(\delta(k_0 - k) + \delta(k_0 + k)\right)
\]
\[ F \sin \left( \frac{2\pi}{N} k_0 x \right) = \sum_{x=0}^{N-1} \sin \left( \frac{2\pi}{N} k_0 x \right) e^{-i \left( \frac{2\pi}{N} k x \right)} \]
\[ = \sum_{x=0}^{N-1} \frac{1}{2i} \left( e^{i k_0 x} - e^{-i k_0 x} \right) e^{-i \frac{2\pi}{N} k x} \]
\[ = -\frac{Ni}{2} \left( \delta(k_0 - k) - \delta(k_0 + k) \right) \]

**Inverse Fourier transform**

Example 4 tells us what the inverse of the Fourier transform matrix \( F \) is. Noting that
\[
\frac{1}{N} \sum_{u=0}^{N-1} e^{-i \frac{2\pi}{N} k_1 u} e^{i \frac{2\pi}{N} k_2 u} = \begin{cases} 
1, & \text{if } k_1 = k_2 \\
0, & \text{if } k_1 \neq k_2.
\end{cases}
\]
we see immediately that
\[
\frac{1}{N} F F^T = I
\]
where \( F \) is the matrix containing the complex conjugate of elements of \( F \). Since \( F \) is symmetric \( F \) is also symmetric, and so
\[
\frac{1}{N} F F = I.
\]
Thus
\[
F^{-1} = \frac{1}{N} F.
\]

**Convolution Theorem**

Finally, I prove one more important property of Fourier transforms which is known as the *Convolution Theorem*: For any two functions \( I(x) \) and \( h(x) \)
\[
F( I(x) * h(x) ) = F I(x) \cdot F h(x).
\]
We will use heavily in the rest of the course.

To prove this theorem, we need to handle the finite range of \( x \) values on which \( I(x) \) and \( h(x) \) is defined, since the convolution will require that we index values outside of this range. There are two ways to handle it. One is to define the summations from \(-\infty\) to \( \infty \). The other is to assume that the functions are periodic, i.e. \( I(x) = I(x + mN) \) and \( h(x) = h(x + mN) \) for any integer \( m \) and define the summation from 0 to \( N - 1 \).
Proof:

\[
F \, I \ast h(x) = \sum_{x=0}^{N-1} e^{-i \frac{2\pi}{N} kx} \sum_{x'=0}^{N-1} I(x-x') h(x'), \quad \text{by definition}
\]

\[
= \sum_{x'=0}^{N-1} h(x') \sum_{x=0}^{N-1} e^{-i \frac{2\pi}{N} kx} I(x-x'), \quad \text{by switching order of sums}
\]

\[
= \sum_{x'=0}^{N-1} h(x') \sum_{u=0}^{N-1} e^{-i \frac{2\pi}{N} k(u+x')} I(u), \quad \text{where } u = x - x'
\]

\[
= \sum_{x'=0}^{N-1} h(x') e^{-i \frac{2\pi}{N} kx'} \sum_{u=0}^{N-1} e^{-i \frac{2\pi}{N} ku} I(u)
\]

\[
= \hat{h}(k) \, \hat{I}(k)
\]