Exercises 3 COMP 423 Winter 2008

Arithmetic coding

1. Consider a first order stationary Markov model with the following probabilities:

$$p(X_1) = \begin{bmatrix} \frac{5}{8} \\ \frac{3}{8} \end{bmatrix} \qquad p(X_{i+1} \mid X_i) = \begin{bmatrix} \frac{7}{8} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{4} \end{bmatrix}$$

such that, in the conditional probability matrix, the element in row j and column k is $p(X_{i+1} = A_j \mid X_i = A_k)$.

Consider a sequence:

$$\vec{x} = A_1\ A_2\ A_2\ A_1$$

   (a) What is $p(\vec{x})$, the probability of the sequence $\vec{x}$?

   (b) What is $pred(\vec{x})$?

   (c) What is the tag $T(\vec{x})$?

2. Consider a sequence $X_1, X_2, , \ldots$ of integer valued random variables, such that:

$$p(X_{j+1} - X_j = -1\ ) = \frac{3}{8}$$
$$p(X_{j+1} - X_j = 0\ ) = \frac{1}{4}$$
$$p(X_{j+1} - X_j = 1\ ) = \frac{3}{8}$$

Assume $X_0 = 0$. Thus, $X_1 \in \{-1, 0, 1\}$, $X_2 \in \{-2, -1, 0, 1, 2\}$, etc.

Suppose the differences, $X_{j+1} - X_j$, were encoded using an arithmetic coder. Let

$$A_1 = -1, \quad A_2 = 0, \quad A_3 = 1$$

be the symbols representing the values of the differences.. Let the codeword for a particular sequence $\vec{x}$ begin with the eleven bits

$$00110101001\ldots$$

What is $\vec{x}$? Give your answer up to as many elements in the sequence as possible.

3. Let (X_1, X_2, X_3, X_4) be a sequence of four independent identically distributed random variables that take values from alphabet $\{\ 0,\ 1\ \}$ such that:

$$\begin{bmatrix} p(\ X_i = 0\ ) \\ p(\ X_i = 1\ ) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$$

Consider a particular sequence:

$$\vec{x} = (0,\ 1,\ 1,\ 1)$$

(a) What is $pred(\vec{x})$?

(b) What is $p(\vec{x})$ i.e. the probability of the sequence $\vec{x}$?

(c) What is $T(\vec{x})$ i.e. the tag of $\vec{x}$?

(d) What is the codeword $C(\vec{x})$?

(e) For the above probabilities, list *all* the sequences $\vec{x}$ of length $n = 4$ such that the codeword length $\lambda(\vec{x})$ is less than or equal to 4.

Predictive and transform coding

1. Consider the sequence
$$(3.4,\ 5.2,\ 3,\ 1,\ -1.5,\ -2,\ -0.7\ )\ .$$
To encode this sequence using differential coding and using the model $\delta_j = x_j - a\hat{x}_{j-1}$, what would be a suitable constant a ? (You do not need to encode the sequence.)

   What general observation can you make about how a depends on the statistics of a sequence?

2. Suppose we have samples of a pair of random variables (X_1, X_2):
$$(1,2),\ (2,2),\ (-1,3),\ (1,2),\ (2,2),\ (-1,1),\ (1,0),\ (-1,0)\ .$$
   (a) What 2×2 transform could be applied to remove correlations between the (X_1, X_2) ?

   (b) If only integer operations could be used by the decoder, what 2×2 (integer valu ed) transform matrix could be sent to the decoder and what would be an appropriate (integer) value of the quantization width Δ ?

3. Consider an audio signal that is represented by 16 bits per sample. Suppose we want to *losslessly* encode such a signal.

   (a) If we encode the samples independently, how many codewords are needed ?

   (b) If we encode the difference between each sample and the previous sample, how many codewords are needed ?

   (c) Under what conditions would you expect (b) to produce a shorter average code length than (a) ?

4. The video compression method described in class was lossless. We can turn it into lossy method by quantizing the intensity differences between each block of the current frame and the intensities predicted from pixels in a nearby frame.

   (a) Specify how to find the offset vector (v_x, v_y) by defining a mathematical expression to be minimized. Your method must ensure that errors do not accumulate.

   (b) Let the display order be a sequence of frames with repeated pattern IBPB. Give the order in which the frames are encoded and decoded. Assume 8 frames, and so your answer is an ordering of the numbers from 1 to 8.

(c) The intensities in B frame blocks may be predicted either from the previous or the next P or I frames, and so one bit is needed per block to distinguish which.

It sometimes fails to happen, however, that every block in every B frame corresponds to a shifted set of pixels in the previous or next I or P frame. Give two distinct reasons for this failure, and suggest a more general scheme to encode such B frame blocks that allows for these failures.

5. The Yule-Walker equations were introduced in the context of lossy linear predictive coding (see end of lecture 26). Here they are for the case $m = 4$:

$$\begin{bmatrix} R(1) \\ R(2) \\ R(3) \\ R(4) \end{bmatrix} = \begin{bmatrix} R(0) & R(1) & R(2) & R(3) \\ R(1) & R(0) & R(1) & R(2) \\ R(2) & R(1) & R(0) & R(1) \\ R(3) & R(2) & R(1) & R(0) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

(a) How are the a_i's used by the encoder and decoder?

(b) The equations are derived by minimizing a total least-squares error. Write out a mathematical expression for this error. Define any constants or variables in your expression.

(c) One can always obtain a lower total least-squares error by increasing m. Why?

(d) Despite the claim of (b), one often does *not* obtain better lossy compression when one increases m. Why not?

6. (a) Consider a lossy differential coder that uses a mid-rise quantizer with $N = 6$ and $\Delta = 2$. Assuming that the first element is encoded with no loss (no error), what will be the decoder's estimate of the sequence:

$$55,\ 58,\ 65,\ 71,\ 64,\ 61,\ \ldots$$

Be sure to explain all steps in your answer.

Hint: Granular error intervals for a mid-rise quantizer are of the form $[\ (l-1)\ \Delta,\ l\ \Delta\ )$.

(b) Assume a stationary sequence of integer valued random variables with mean zero, and with high positive correlation between nearby samples. Let the sequence be encoded using lossy differential coding with a mid-tread quantizer with $N = 3$.

Describe the errors in the decoder's estimate of a typical sequence, when:

- the chosen Δ is *too small*, and so overload errors almost always occur;
- the chosen Δ is *too large*, and so overload errors almost never occur.

Give an example to illustrate your argument in each case.