Questions

1. Use mathematical induction to prove that, for any $n \geq 1$

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}.$$

2. Use mathematical induction to prove that, for all $n \geq 1$,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

3. Use mathematical induction to prove that, for all $n \geq 1$,

$$1 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$
Answers

1. The base case is easy. Substitute \( n = 1 \) and we get \( 1 = 1 \) which is true.

   For the induction step, we hypothesize that
   \[
   \sum_{i=0}^{k-1} x^i = \frac{x^k - 1}{x - 1}
   \]
   for \( k \geq 0 \), and we want to show it follows from this hypothesis that
   \[
   \sum_{i=0}^{k} x^i = \frac{x^{k+1} - 1}{x - 1}.
   \]

   Take the left side of the last equation, and rewrite it:
   \[
   \sum_{i=0}^{k} x^i = \sum_{i=0}^{k-1} x^i + x^k
   \]
   \[
   = \frac{x^k - 1}{x - 1} + x^k, \quad \text{by induction hypothesis}
   \]
   \[
   = \frac{x^k - 1}{x - 1} + x^k \left(\frac{x - 1}{x - 1}\right)
   \]
   \[
   = \frac{x^{k+1} - 1}{x - 1}
   \]
   which is what we wanted to show.

2. The base case of \( n_0 = 1 \) is obvious, since there is only a single term on the left hand side, i.e. \( 1 = 1^2 \). The induction hypothesis is the statement \( P(k) \):
   \[
   P(k) \equiv "1 + 3 + 5 + \cdots + (2k - 1) = k^2 "
   \]

   To prove the induction step, we show that if \( P(k) \) is true, then \( P(k + 1) \) must also be true. As usual, we take the left side of the equation of \( P(k + 1) \):
   \[
   \sum_{i=1}^{k+1} (2i - 1) = 2(k + 1) - 1 + \sum_{i=1}^{k} (2i - 1)
   \]
   \[
   = 2(k + 1) - 1 + k^2, \quad \text{by the induction hypothesis}
   \]
   \[
   = 2k + 1 + k^2
   \]
   \[
   = (k + 1)^2.
   \]

   Thus, the induction step is also proved, and so we’re done.
3. The base case is trivially obvious since $1^3 = 1^2$.

To prove the induction step, we write

\[ 1 + 2^3 + 3^3 + \cdots + k + (k + 1)^3 \]
\[ = (1 + 2 + \cdots + k)^2 + (k + 1)^3 \] by the induction hypothesis
\[ = \left( \frac{k(k + 1)}{2} \right)^2 + (k + 1)^3 \]
\[ = \left( \frac{k^2}{4} + (k + 1) \right) \cdot (k + 1)^2 \]
\[ = \frac{1}{4} (k^2 + 4k + 4) \cdot (k + 1)^2 \]
\[ = \frac{1}{4} (k + 2)^2 \cdot (k + 1)^2 \]
\[ = \left\{ \frac{1}{2} (k + 2)(k + 1) \right\}^2 \]
\[ = (1 + 2 + 3 + \cdots + k + (k + 1))^2. \]