Questions

In the following questions, we sometimes write $t(n) \in O(g(n))$ instead of $t(n)$ is $O(g(n))$, where "$\in$" simply means "is a member of the set of functions that is $O(g(n))$."

1. Is $\cos(n) \in O(n)$?
2. Is $\cos(n) \in O(1)$?
3. Is $n! \in O((n+2)!)$?
4. Is $(n+2)! \in O(n!)$?
5. Is $9^n \in O(12^n)$?
6. Is $12^n \in O(9^n)$?
7. Let
$$t(n) = \sum_{i=0}^{n} 3^i.$$
Show that $t(n)$ is $O(3^n)$.
8. In lecture 9, I showed that for all n, $Fib(n) \le 2^n$. It follows trivially that $Fib(n)$ is $O(2^n)$.
Use induction to prove a tighter bound, namely $Fib(n)$ is $O(\ (\frac{7}{4})^n\ )$.
9. Use induction to prove a lower bound: $Fib(n) \in \Omega((\frac{3}{2})^n)$.
10. Let $t(n) = (n+8)^{1.3} + 3n + 5$. Prove that $t(n)$ is $O(n^{1.3})$.
11. Let $t(n) = n \log n$. Prove that $t(n)$ is $\Omega(\log(n!))$.
12. Let $t(n) = 5n^2 + 3n + 4$. Suppose we wanted to show that $t(n)$ is $O(n^2)$. Here is an *incorrect* proof:
We would like to show that, for n sufficiently large,
$$5n^2 + 3n + 4 \le cn^2.$$
Divide both sides by n^2 gives
$$5 + \frac{3}{n} + \frac{4}{n^2} \le c.$$
Taking limit as $n \to \infty$ gives $5 \le c$. So we choose $c = 5$ and we are done.
Why is this proof incorrect ?
13. What is wrong with the following proof that "$t(n) = n + 2$ is $O(n)$" ?
[I have written line numbers (1)-(3) because I discuss each line in the solutions.]

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(Incorrect) Proof:   n+2 <= cn                    (1)
                     n+2 <= 2n                    (2)
                       2 <= n                     (3)
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14. Let $t(n) = \lfloor \log n \rfloor - 1$. Prove that is $t(n)$ is $\Omega(\log n)$
15. Is it possible to have functions $t(n)$ and $g(n)$ such that $t(n)$ is $O(g(n))$ and $t(n)$ is $\Omega(g(n))$?
16. Let $t(n)$ be $O(g(n))$. Does this imply that $t(n)$ is NOT $\Omega(g(n))$?
17. Show $t(n) = \log(\log n)$ is $O(\log n)$.
18. If $t(n) \in O(g(n))$, may we conclude that $g(n) \in \Omega(t(n))$?
19. If $t(n) \in \Omega(g(n))$, may we conclude that $g(n) \in O(t(n))$?
20. Show 2^n is $O(n!)$.
21. Show $n!$ is *not* $O(2^n)$.

Answers

1. $\cos(n) \leq 1 \leq n$ when $n \geq 1$, so yes. (Take $c = 1, n_0 = 1$.)

2. $\cos(n) \leq 1$, so yes. Again, take $c = 1, n_0 = 1$.

3. We need to show there exists $c, n_0 \geq 0$ such that

$$n! \leq c(n+2)(n+1) \cdot n!$$

for all $n \geq n_0$. Letting $c = 1$ and $n_0 = 1$ and dividing by $n!$, we now need to show that $1 \leq (n+1)(n+1)$. But this is obviously true. Thus $n! \in O((n+2)!)$

4. Here we need to find a $c, n_0 \geq 0$ such that $(n+2)(n+1) \cdot n! \geq cn!$ for all $n \geq n_0$. Dividing by $n!$, we now need to show $(n+1)(n+2) \leq c$ for all $n \geq n_0$. But this is clearly false, since the left side grows without bound as n grows.

   To get specific values (instead of just using a limit argument), choose positive c and n_0. Note that $(n+1)(n+2) > (n+1)^2$ and solve $(n+1)^2 = c$ for n, namely $\sqrt{c} - 1$. Take $n > max(\sqrt{c} - 1, n_0$ gives the contradiction. Thus, $(n+2)! \notin O(n!)$ i.e. $(n+2)!$ is not $O(n!)$.

5. Yes, and it is trivial. Since $9 < 12$, it follows that $9^n < 12^n$ and so $c = 1$ and $n_0 = 1$ does the job.

6. We want to show there exists $c, n_0 \geq 0$ such that $12^n \leq c9^n$ for all $n \geq n_0$. But, letting $\iff$ denote "if and only if", we have

$$\begin{aligned} & 12^n \leq c9^n \\ \iff \quad & \log(12^n) \leq \log(c9^n) \\ \iff \quad & n \log 12 \leq \log c + n \log 9 \\ \iff \quad & n \leq \frac{\log c}{\log \frac{12}{9}} \end{aligned}$$

   But this inequality cannot be true for all $n \geq n_0$ obviously. Thus, $12^n \notin O(9^n)$.

7. Recall the formula for a geometric series

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}.$$

   Then,

$$\sum_{i=1}^{n} 3^i = \frac{3^{n+1} - 1}{3 - 1} = \frac{3}{2}(3^n - \frac{1}{3})$$

   which is $O(3^n)$, i.e. take $c = \frac{3}{2}$ and $n_0 = 1$.

8. We need to find an n_0 and c such that, for all $n \geq n_0$, $F(n) \leq c(\frac{7}{4})^n$.

   Try $c = 1$. The base case is trivial since $F(0) = 0 < (\frac{7}{4})^0$ and $F(1) = 1 < \frac{7}{4}$. So let's hypothesize that $F(n) \leq (\frac{7}{4})^n$ for all n up to some $k \geq 1$ and see if it follows for $n = k + 1$.

$$\begin{aligned} F(k+1) &= F(k) + F(k-1) \\ &\leq (\frac{7}{4})^k + (\frac{7}{4})^{k-1} \text{ by the induction hypothesis,} \\ &= (\frac{7}{4} + 1)(\frac{7}{4})^{k-1} \end{aligned}$$

   But it is easy to verify that $\frac{7}{4} + 1 < (\frac{7}{4})^2$ and so (from the induction hypothesis) we get

$$\begin{aligned} F(k+1) &\leq (\frac{7}{4})^2 (\frac{7}{4})^{k-1} \\ &= (\frac{7}{4})^{k+1}. \end{aligned}$$

   This proves the induction step, and so we are done.

9. We need to find an n_0 and c such that $F(n) \geq c(\frac{3}{2})^n$ for all $n \geq n_0$.

   Let's first establish a base case. We can't have a base case for $n = 0$ since $F(0) = 0$ and so it will be impossible for $F(0) \geq c(\frac{3}{2})^0$ for $c > 0$. Instead, we try to find a c and use the base case(s) $n = 1, 2$. If we let $c = (\frac{2}{3})^2$, then indeed we have $F(n) \geq c(\frac{3}{2})^n$ for $n = 1, 2$. So let's try using that c and proving the induction step.

   We assume the induction hypothesis, namely we assume that $F(n) \geq c(\frac{3}{2})^n$ for $n = k - 1, k$. We want to show it follows that $F(k + 1) \geq c(\frac{3}{2})^{k+1}$.

$$\begin{aligned} F(k+1) &= F(k) + F(k-1) \\ &\geq c(\frac{3}{2})^k + c\,(\frac{3}{2})^{k-1} \text{ by induction hypothesis} \\ &= c\,(\frac{3}{2} + 1)\,(\frac{3}{2})^{k-1} \\ &> c\,(\frac{3}{2})^2(\frac{3}{2})^{k-1}, \text{ since } \frac{5}{2} > \frac{9}{4} \\ &= c\,(\frac{3}{2})^{k+1} \end{aligned}$$

   Thus, both the base case and induction step are proved and so we are done.

10. We need to show there exists two positive constants c, n_0 such that, for all $n \geq n_0$,

$$(n + 8)^{1.3} + 3n + 5 \leq cn^{1.3}.$$

$$\begin{aligned} (n+8)^{1.3} + 3n + 5 &\leq (2n)^{1.3} + 3n + 5, \quad \text{if } n \geq 8 \\ &< 4n^{1.3} + 3n^{1.3} + 5n^{1.3}, \quad \text{since } 2^{1.3} < 2^2 = 4 \\ &= 12n^{1.3} \end{aligned}$$

   So, take $n_0 = 8$ and $c = 12$.

11. We need to show there exist two positive constants c, n_0 such that, for all $n \geq n_0$,

$$n \log n \geq c \log(n!).$$

Since $n \log n = \log(n^n)$, and since $\log x$ is monotonically increasing, it is enough for us to show that there exist n_0, c such that, for all $n \geq n_0$,

$$n^n \geq cn!$$

But this is trivial. Take c = 1, and note $\frac{n^n}{n!} \geq 1$ since both numerator and denominator have n terms and ratio of corresponding terms is greater than or equal to 1 (so the product of the ratios is greater than or equal to 1).

12. The proof is incorrect because I needed to find a *finite* n_0 such that the inequality holds for all $n \geq n_0$. But no such finite n_0 exists. (If I had taken a $c > 5$ rather than $c = 5$, then I could have found such an n_0.)

13. The main problem with the proof is that the reader has no idea what the writer is assuming and what he is proving. For example, statement (1) doesn't tell the reader whether this statement is given, or whether it is something the writer is trying to prove. Even if the writer were to state that (1) is what the writer is trying to prove, the writer also needs to give quantifiers for c and n, namely he wants prove that this statement holds *for some c and* n_0 and *for all* $n >= n_0$ (not for all c, n_0 and some n, etc.)

Suppose we were to correct line (1) as I just discussed. What about line (2) ? Line (2) would only follows from line (1) if the writer wrote explicitly that $c = 2$, which he didn't do. So again, it is confusing.

(3) is also a problem. What is the relationship between (2) and (3)? Does (3) follow from (2)? Does (2) follow from (3)? Are (2) and (3) equivalent ?

Here is a correct proof, for the record:

We want to show that there exists c and n_0 such that $n + 2 <= cn$ for all $n >= n_0$. i.e. we state what we are trying to prove.

Observe that $n + 2 <= 2n$ for all $n \geq 2$, so we can take $c = 2$ and $n_0 = 2$, and we are done.

14. We need to show there exist two positive constants c, n_0 such that, for all $n \geq n_0$,

$$\lfloor \log n \rfloor - 1 \geq c \log n.$$

But

$$\lfloor \log n \rfloor - 1 \geq \log n - 2.$$

Let's guess at a constant $c < 1$ and show we can find an n_0 to get what we want. Guess $c = \frac{1}{2}$, and check if there is an n_0 such that, for all $n \geq n_0$,

$$\log n - 2 \geq \frac{1}{2} \log n \ .$$

But this inequality is the same as the following:

$$\frac{1}{2}\log n \geq 2 \ .$$

which is true when $n \geq 16$. So take $n_0 = 16$ and we are done.

15. YES, YES, YES. Indeed most of the examples that you will see in this course have this property. The only exception in these exercises is examples like $n!$ and $Fib(n)$ where we state bounds in terms of polynomials or exponential functions.

16. No, "$t(n)$ is $O(g(n))$" does not imply that "$t(n)$ is NOT $\Omega(g(n))$," since it is possible for a $t(n)$ to be both $O(g(n))$ and $\Omega(g(n))$. See Question 3. (You should appreciate that there are many such examples.)

17. This may be "obvious" but let's just go through the proof anyhow to make sure we see why it is obvious.

    We want to show there exists c, n_0 such that, for all $n \geq n_0$,

    $$\log\log n \leq c\log n \ .$$

    Equivalently (using the fact that $x < y$ if and only if $2^x < 2^y$), we want to show

    $$\log n \leq 2^{c\log n} = 2^{\log(n^c)} = (n^c),$$

    or equivalently (using the same fact as above)

    $$n \leq 2^{(n^c)}.$$

    Take $c = 1$ and $n_0 = 0$. Indeed, $n < 2^n$ for all integers $n \geq 1$).

18. The answer is yes, and here is the proof. If $t(n) \in O(g(n))$, then there exists a constant c and n_0 such that, for all $n \geq n_0$,

    $$t(n) \leq cg(n)$$

    or equivalently

    $$g(n) \geq \frac{1}{c}t(n).$$

    Hence the constants n_0 and $\frac{1}{c}$ exist for $g(n)$ is $\Omega(t(n))$.

19. The answer is yes, and the proof is exactly as in the previous question. If $t(n) \in \Omega(g(n))$, then there exists a constant c and n_0 such that, for all $n \geq n_0$,

    $$t(n) \geq cg(n)$$

    or equivalently

    $$g(n) \leq \frac{1}{c}t(n).$$

    Hence the constants n_0 and $\frac{1}{c}$ exist for $g(n)$ is $O(t(n))$.

20. We want to show that there exist two constants $c > 0$ and $n_0 > 0$ such that, for all $n > n_0$,

$$2^n \ \leq \ c\, n!$$

or, equivalently,

$$\frac{2}{n} \cdot \frac{2}{n-1} \cdot \frac{2}{n-2} \cdots \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{2}{2} \cdot \frac{2}{1} \leq c.$$

On the left side, the numerator and denominator have n terms each. We pair them up and note that numerator terms are all less than or equal to their corresponding denominator terms, except for the last pair $(\frac{2}{1})$. We take the last pair to the other side,

$$\frac{2}{n} \cdot \frac{2}{n-1} \cdot \frac{2}{n-2} \cdots \frac{2}{4}\frac{2}{3} \cdot \frac{2}{2} \leq \frac{c}{2}.$$

Now we see that if we let $c = 2$ and $n_0 = 1$, then this inequality indeed holds for all $n \geq n_0$ since the right side is 1 and the left side is a product of terms that are each greater than or equal to 1.

21. Take any two positive constants c, n_0. We want to show that there exists an $n > n_0$ such that

$$n! \ > \ c\, 2^n$$

or equivalently, following the logic from the previous question, we want to find $n > n_0$ such that

$$\frac{n}{2} \cdot \frac{n-1}{2} \cdot \frac{n-2}{2} \cdots \frac{4}{2} \cdot \frac{3}{2} \cdot \frac{2}{2} \cdot \frac{1}{2} > c.$$

Since c can be any positive constant, it may require quite a large n before the product on the left side is greater than c.

On the left side, both the numerator and denominator have n terms and if we pair them up then the numerator terms are all greater than or equal to the denominator terms except for the last one $(\frac{1}{2})$.

By inspection, if $n > max(n_0, 2c, 4)$, then the above inequality holds. The reason is that:

$$\frac{n}{2} \cdot \frac{n-1}{2} \cdot \frac{n-2}{2} \cdots \frac{4}{2} \cdot \frac{3}{2} \cdot \frac{2}{2} \cdot \frac{1}{2} \ > \ \frac{2c}{2} \cdot \frac{n-1}{2} \cdot \frac{n-2}{2} \cdots \frac{4}{2} \cdot \frac{3}{2} \cdot \frac{2}{2} \cdot \frac{1}{2} \ > \ c$$

and the expression in the middle is c multiplied by a product of terms that are all greater than 1, multiplied by $\frac{4}{2} \cdot \frac{3}{2} \cdot \frac{2}{2} \cdot \frac{1}{2}$ which is greater than 1.