COMP 250 - Assignment 2 - Question 2 (Solutions)
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General Notes

- The marking scheme was set by M. Langer.
- The grading was done by Jonathan.
- For questions A and B, you need to find provide values of c and n_0 that satisfy the definitions of $O(\ )$ and $\Omega(\ )$. You also need to argue convincingly that these constants meet the definitions, i.e. you need to give a proof. You may have a valid c, n_0 pair, but if your proof doesn't show *why* this is a valid pair, then you will not get full points.

  Note: the whole point here for you to demonstrate that you understand the definitions and you know how to prove the truth or falsity of statements that use the definitions!
- There are 10 points for each question, for a total of 50 points. For each question, we have included a marking scheme.

A.

We want to show there exists a $c > 0$ and $n_0 \geq 1$ such that, for all $n \geq n_0$,

$$\sqrt{31n + 12n\log n + 57} < c\sqrt{n}\log n. \tag{1}$$

But

$$\begin{aligned}
\sqrt{31n + 12n\log n + 57} &< \sqrt{31n\log n + 12n\log n + 57n\log n}, \text{ when } \; n > 2 && (2)\\
&= \sqrt{100n\log n} && (3)\\
&\leq 10\;\sqrt{n}\;\sqrt{\log n} && (4)\\
&< 10\sqrt{n}\log n && (5)
\end{aligned}$$

where the last line follows from the fact that $\sqrt{x} < x$ when $x > 1$. So, take $n_0 = 2$ and $c = 5$.

Marking scheme

Here are the elements of the solution. We gave 2-3 points for each.

- State what needs to be proved, i.e. apply the definition of big O to this problem (Eq. 1).
- Observe a concrete upper bound, for a particular n_0 (Eq. 4).
- Relax to a slightly looser upper bound (Eq. 5).
- Choose a c that is based on this upper bound.

For students who were able to get a c and n_0, we took off 1 point for each incorrect or ambiguous statement in the solution.

B.

Using the definition of $\Omega(\ )$, we want to show there exists a $c > 0$ and $n_0 \geq 1$ such that, for all $n \geq n_0$,

$$(n - 3\log n)^{1.6} + 5n^{1.5} + 7 \ \geq \ cn^{1.6}. \tag{6}$$

First notice that

$$(n - 3\log n)^{1.6} + 5n^{1.5} + 7 \ \ \geq (n - 3\log n)^{1.6}, \quad \text{for } n \geq 1 \tag{7}$$

and so it will be enough to find a suitable lower bound for the expression on the right side, in particular, we want to show that there exists a $c > 0$ and $n_0 \geq 1$ such that, for all $n \geq n_0$,

$$(n - 3\log n)^{1.6} \geq cn^{1.6}.$$

Take both sides to the power $\frac{1}{1.6}$, gives

$$n - 3\log n \geq c^{1/1.6}n \tag{8}$$

Let $C = c^{1/1.6}$, so we want:

$$(1 - C)n \geq 3\log n. \tag{9}$$

Try $C = \frac{1}{2}$ (or $c = 2^{-1.6}$) and we see that the inequality indeed holds for all $n \geq 32$. So we take $n_0 = 32$ and $c = 2^{-1.6}$.

Marking scheme

Here are the elements of the solution. The general guideline was to give 2 points for each.

- State what they want to prove (Eq. 6).
- Drop the term with power 1.5 and the constant term. (Eq. 7).
- Re-express without power 1.6 (Eq. 8).
- Manipulate algebraicically to Eq. 9.
- Solve for C and then c.

We subtracted points if the solution was not clearly presented. (This was an assignment, not an exam, so we expected an appropriate level of neatness here.)

C.

Let $n = 2^k$ and so $\log n = k$. Then,

$$\begin{aligned}
t(n) &= 5\ t(\frac{n}{2}) + n && (10)\\
&= 5(5t(\frac{n}{2^2}) + n/2) + n && (11)\\
&= 5^2 t(\frac{n}{2^2}) + n\frac{5}{2} + n && (12)\\
&= 5^2(\ 5\ t(\frac{n}{2^3}) + \frac{n}{2^2}) + n\frac{5}{2} + n && (13)\\
&= 5^3\ t(\frac{n}{2^3}) + n\ \frac{5^2}{2^2} + n\frac{5}{2} + n && (14)\\
&= 5^k t(\frac{n}{2^k}) + n(\frac{5}{2})^{k-1} + \cdots + n\frac{5}{2} + n && (15)\\
&= t(1)5^{\log n}\ +\ n \sum_{i=0}^{\log n - 1} (\frac{5}{2})^i && (16)\\
&= t(1)5^{\log n}\ +\ n\ ((\frac{5}{2})^{\log n} - 1)\ /\ (\frac{5}{2} - 1) && (17)
\end{aligned}$$

You can leave it like that. Or you can follow the manipulation at the end of lecture 16 to get

$$t(n) = t(1)n^{\log 5}\ +\ \frac{2}{3}(n^{\log 5} - n) \tag{18}$$

Marking scheme

We generally gave 2 points for each of the following steps:

- result of first backward substitution (eq. 12)
- result of second backward substitution (eq. 14)
- general substitution substitution (eq. 15)
- rewrite geometric series as a summation (eq. 16)
- simplify the expression, by removing the summation (eq. 17)

We gave the full 10 points if you left it like this, but note that it could be pushed a bit further, namely eq. 18.

D.

$$\begin{aligned} t(n) &= t(\frac{n}{2}) + 3\log n && (19)\\ &= (t(\frac{n}{2^2}) + 3\log(\frac{n}{2})) + 3\log n && (20)\\ &= (t(\frac{n}{2^3}) + 3\log(\frac{n}{2^2})) + 3\log(\frac{n}{2}) + 3\log n && (21)\\ &= t(\frac{n}{2^k}) + 3\log(\frac{n}{2^{k-1}}) + ... + 3\log(\frac{n}{2}) + 3\log n && (22)\end{aligned}$$

Let $n = 2^k$.

$$\begin{aligned} t(n) &= t(1) + 3\ (\log 2 + \log 4 + ... + \log\frac{n}{2} + \log n) && (23)\\ &= t(1) + 3\ (1 + 2 + 3 + ... + \log n) && (24)\\ &= t(1) + 3\ \sum_{i=1}^{\log n} i && (25)\\ &= t(1) + 3\ \log(n)\frac{(\log(n)+1)}{2} && (26)\end{aligned}$$

Marking scheme

We generally gave 2 points for each of the following elements of the solution.

- first backward substitution (eq. 20)
- second backward substitution (eq. 21)
- full substitution (eq. 22)
- simplification to eq. 24
- sum of arithmetic series (recall Gauss at 4 years old)

E.

$$\begin{aligned}
t(n) &= 2t(\frac{n}{2}) + n^2 && (27)\\
&= 2[2t(\frac{n}{2^2}) \ + (\frac{n}{2})^2] + n^2 && (28)\\
&= 2^2t(\frac{n}{2^2}) \ + \frac{n^2}{2} + n^2 && (29)\\
&= 2^2[2t(\frac{n}{2^3}) + (\frac{n}{2^2})^2] + \frac{n^2}{2} + n^2 && (30)\\
&= 2^3t(\frac{n}{2^3}) + \frac{n^2}{4} + \frac{n^2}{2} + n^2 && (31)\\
&= 2^kt(\frac{n}{2^k}) + \frac{n^2}{2^{k-1}} + \frac{n^2}{2^{k-2}} + ... + \frac{n^2}{2} + n^2 && (32)\\
& && (33)
\end{aligned}$$

Let $2^k = n$, and so $k = \log n$.

$$\begin{aligned}
t(n) &= nt(1) + n^2 \sum_{i=0}^{\log(n)-1} \frac{1}{2^i} && (34)\\
&= nt(1) + n^2((\frac{1}{2})^{\log n} - 1)/(\frac{1}{2} - 1) && (35)\\
&= nt(1) + 2n^2(1 - \frac{1}{n}) && (36)\\
&= nt(1) + 2n^2 - 2n && (37)
\end{aligned}$$

Here is is somewhat surprising that the answer is $O(n^2)$. In eyeballing the given recurrence, one might have guessed that there would be a further dependence on $\log n$. But that is not so.

Marking scheme

We generally gave 2 points for each of the following elements of the solution.

- first backward substitution (eq. 27)
- second backward substitution (eq. 29)
- full substitution (eq. 30)
- rewrite geometric series as a summation (eq 31)
- simplify the summation, by removing the summation (eq. 33)