COMP 250

lecture 19

mathematical induction
The course so far has been mostly object orientated programming (in Java).

From here on, it will be mostly data structures, algorithms, and computational complexity (mathematical analysis).

Many of the algorithms will use recursion. Recursion is closely related to mathematical induction. We will cover these topics next.
Recall lecture 0 slide 11:

Math Prerequisites

CEGEP level math (Cal 1):

\[ 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2} \]

\[ 1 + x + x^2 + x^3 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} \]

\[ \log(ab) = \log a + \log b \]
How to prove the following *statement*?

For all $n \geq 1$,

$$1 + 2 + 3 + \ldots + (n - 1) + n = \frac{n(n+1)}{2}.$$  

By “proof”, we mean a formal logical argument that convincingly shows the statement is true.

You’ve probably seen before the proof shown on the next two slides.
Consider the sum: \( 1 + 2 + \ldots + (n - 1) + n \).

If \( n \) is even, then we can group into \( \frac{n}{2} \) pairs:

\[
1 + 2 + \ldots + \frac{n}{2} + \left( \frac{n}{2} + 1 \right) \ldots + (n - 1) + n
\]

Each pair adds up to \( n + 1 \) and there are \( \frac{n}{2} \) pairs.

Adding them up gives \( \frac{n}{2} (n+1) \).
If $n$ is odd, then, $n-1$ is even. So,

$$1 + 2 + \ldots + (n - 1) + n$$

$$= \left(\frac{n-1}{2}\right)n + n$$

$$= \left(\frac{n-1}{2} + 1\right)n$$

$$= \left(\frac{n+1}{2}\right)n$$ which is the same formula as before.
Mathematical Induction

Consider a statement of the form:

“For all \( n \geq n_0, \ P(n) \) is true” where \( n_0 \) is some constant, and \( P(n) \) is some proposition that has a value true or false which may depend on \( n \).

Mathematical induction is a general technique for proving such a statement.

Note that the statement about infinitely many \( n \)’s!
“For all $n \geq n_0$, $P(n)$ is true.”

For our previous example:

For all $n \geq 1$,

$$1 + 2 + \ldots + (n - 1) + n = \frac{n(n+1)}{2}$$

is true.

Note in mathematics, one typically does not write the “is true” part of the statement. I am writing it here to emphasize that $P(n)$ can be thought of as a boolean valued expression. Also note we are not treating $n \geq n_0$ as a Boolean valued expression but rather as short hand for “$n$ greater than $n_0$”. 
To prove a statement “For all $n \geq n_0$, $P(n)$ is true” we will use the following analogous concept:

Suppose you have an *infinite* sequence of stepping stones numbered 0, 1, ... and you know how to jump from any stone to the next one, and you start on stone 0 (or more generally $n_0$). Then you will be able to reach all stones (eventually).
To prove a statement “For all $n \geq n_0$, $P(n)$ is true” using mathematical induction, we show:

**Base case:**

$P(n_0)$ is true.

**Induction step:**

For any $k \geq n_0$, if $P(k)$ is true, then $P(k + 1)$ is also true.
The statement “\( P(k) \) is true” is called the **induction hypothesis**.
For any $n_0$, if $P(n_0)$ is true then $P(n)$ is true.

**Base case:**

$P(n_0)$ is true.

**Induction step:**

For any $k \geq n_0$, if $P(k)$ is true then $P(k + 1)$ is true.

If we can prove the base case and induction step (both are true), then we can conclude:

For any $n \geq n_0$, $P(n)$ is true.
Example 1

Statement: For all $n \geq 1$,

$$1 + 2 + 3 + \ldots + (n - 1) + n = \frac{n(n + 1)}{2}$$

Proof (base case, $n = 1$):

?
Example 1

Statement: For all $n \geq 1$,

$$1 + 2 + 3 + \ldots + (n - 1) + n = \frac{n(n + 1)}{2}$$

Proof (base case, $n = 1$):

$$1 = \frac{1(1+1)}{2} \quad \text{(this is true)}$$
The *induction hypothesis*: $P(k)$ is true:

\[
1 + 2 + \ldots + k = \frac{k(k + 1)}{2}
\]

**Base case:**

$P(1)$ is true.

**Induction step:**

For any $k \geq 1$, if $P(k)$ is true, then $P(k + 1)$ is true.
Proof of Induction Step:

\[(1 + 2 + 3 + \ldots + k) + k + 1\]

\[= \frac{k(k+1)}{2} + k + 1 \quad \text{by the induction hypothesis}\]
Proof of Induction Step:

\[
(1 + 2 + 3 + \ldots + k) + k + 1
\]

\[
= \frac{k(k+1)}{2} + k + 1 \quad \text{by the induction hypothesis}
\]

\[
= \left( \frac{k}{2} + 1 \right) (k + 1)
\]

\[
= \frac{1}{2} (k + 2)(k + 1)
\]

Thus, if \( P(k) \) is true then \( P(k + 1) \) is true.
Ambiguities & Possible Confusion

We are using “$a = b$” in two different ways.

• a boolean valued statement, which may be either true or false

• “a equals b” : for example, when we do an algebraic manipulation such as $k(k + 1) = k^2 + k$ as previous slide

When I wrote “$k(k + 1) = k^2 + k$” on the previous slide, I was not saying this is a Boolean expression that is either true of false. Rather I was saying it is true!

It should be clear from the context which usage I mean.
There was a similar ambiguity in Quiz 1 (Fall 2021). How to interpret the expression “38 % x == 2” below?

Let \(38 \% x == 2\). What could be the value of \(x\)? Select all that apply.

- \(12\)
- \(14\)
- \(16\)
- \(18\)

Some students interpreted it as a Java expression (boolean value) and claimed that the sentence didn’t makes sense. They said that I should have written “38 % x equals 2”. (BTW, I gave 0.5 pts to students who interpreted it this way and checked all the boxes.)
Mathematical Induction: Example 2

Prove the following statement:
For all $n \geq 3$, $2n + 1 < 2^n$. 

![Graph showing the growth of $2n + 1$ and $2^n$ for $n = 0, 1, 2, 3$.]
Statement: For all \( n \geq 3 \), \( 2n + 1 < 2^n \) is true.

Note: \( P(n) \) is false for \( n = 1, 2 \).

But that is not a problem. Why not?
Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof (base case, $n = 3$):

?
Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof (base case, $n = 3$):

$2 \times 3 + 1 < 8$ \hspace{1cm} (true)
Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof of Induction Step:

For any $k \geq 3$,

We want to show: if $P(k)$ is true, then $P(k + 1)$ is true.
Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof of Induction Step:

We want to show: if $P(k)$ is true, then $P(k + 1)$ is true.

For any $k \geq 3$,

$2(k + 1) + 1 = ?$

(The left side is $2n + 1$ where $n = k + 1$.)
Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof of Induction Step:

For any $k \geq 3$,

We want to show: if $P(k)$ is true, then $P(k + 1)$ is true.

$2(k + 1) + 1 = 2k + 2 + 1$

trivial calculation
(i.e. not boolean expression that may be either true or false)
Proof of Induction Step:

For any \( k \geq 3 \),

We want to show: if \( P(k) \) is true, then \( P(k + 1) \) is true.

\[
2(k + 1) + 1 = 2k + 2 + 1 < 2^k + 2
\]

by induction hypothesis (where \( k \geq 3 \))
Statement: For all \( n \geq 3 \), \( 2n + 1 < 2^n \).

Proof of Induction Step:

For any \( k \geq 3 \),

We want to show: if \( P(k) \) is true, then \( P(k + 1) \) is true.

\[
2(k + 1) + 1 = 2k + 2 + 1 \\
< 2k + 2 \\
< 2^k + 2^k, \\
= 2^{k+1}
\]

by induction hypothesis (where \( k \geq 3 \))

This inequality is also true for \( k \geq 2 \) but we don’t care because we are trying to prove for \( k \geq 3 \).
Example 3

Statement: For all $n \geq 5$, $n^2 < 2^n$. 
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Base case ($n = 5$):

Induction step:
Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case ($n = 5$):

$25 < 32$

Induction step:

What do we hypothesize? (if ____):

What do we want to show? (then ____)

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Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case ($n = 5$):

\[ 25 < 32 \]

Induction step:

What do we hypothesize? (if ____):

\[ k^2 < 2^k, \text{ where } k \geq 5 \]

What do we want to show? (then____)\

\[ (k + 1)^2 < 2^{k+1} \]
Statement: For all \( n \geq 5 \), \( n^2 < 2^n \).

Base case \((n = 5)\):

\[
25 < 32
\]

Induction step: Let \( k \geq 5 \)

\[
(k + 1)^2 = k^2 + 2k + 1
\]

trivial calculation
Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case ($n = 5$):

$$25 < 32$$

Induction step: Let $k \geq 5$

$$(k + 1)^2 = k^2 + 2k + 1 \quad \text{by induction hypothesis}$$

$$< 2^k + 2k + 1$$
Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case ($n = 5$):

$$25 < 32$$

Induction step: Let $k \geq 5$

$$(k + 1)^2 = k^2 + 2k + 1$$

by induction hypothesis

$$< 2^k + 2k + 1$$

by Example 2 (for $k \geq 3$, so it also holds for $k \geq 5$)

$$< 2^k + 2^k$$

$$= 2^{k+1}$$

which is what we wanted to show.
Example 4: Fibonacci Sequence

0, 1, 2, 3, 5, 8, 13, 21, 34, 55, ....

\[
\begin{align*}
F(0) &= 0 \\
F(1) &= 1 \\
F(n + 2) &= F(n + 1) + F(n), \text{ for } n \geq 0.
\end{align*}
\]

Statement: For all \( n \geq 0 \), \( F(n) < 2^n \)

Let’s prove it by mathematical induction.
\[ F(0) = 0 \]
\[ F(1) = 1 \]
\[ F(n + 2) = F(n + 1) + F(n), \text{ for } n \geq 0. \]

For all \( n \geq 0, \quad F(n) < 2^n \) is true.

Base case(s):

\[ n = 0: \quad 0 < 2^0 \quad \text{yes, this is true} \]
\[ n = 1: \quad 1 < 2^1 \quad \text{yes, this is true} \]
\[ F(0) = 0 \]
\[ F(1) = 1 \]
\[ F(n + 2) = F(n + 1) + F(n), \text{ for } n \geq 0. \]

For all \( n \geq 0 \), \( F(n) < 2^n \) is true.

Induction step:

\[ F(k + 1) = F(k) + F(k - 1) \]
\[ < 2^k + 2^{k-1} \]

by induction hypothesis
(applied twice, \( k > 0 \))
\[ F(0) = 0 \]
\[ F(1) = 1 \]
\[ F(n + 2) = F(n + 1) + F(n), \text{ for } n \geq 0. \]

For all \( n \geq 0 \), \( F(n) < 2^n \)

**Induction step:**

\[ F(k + 1) = F(k) + F(k - 1) \]

\[ < 2^k + 2^{k-1} \]

\[ < 2^k + 2^k \]

\[ = 2^{k+1} \]

which is what we wanted to show.
Exercises

1. Use mathematical induction to prove that, for any $n \geq 1$

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}.$$ 

2. Use mathematical induction to prove that, for all $n \geq 1$,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$