WSEAS Trans. Circuits and Systems, Vol.3, No.3, April 2004, pp.640-646.

Design of Robust SISO Controllers for Stable Plants Using FIR Q Parameters

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Abstract: The tradeoff between performance and robustness in single-input single-output linear time-invariant stable plants represents a fundamental aspect of robust control design. We use a theorem that clearly exposes such a tradeoff between the maximum magnitude of the additive plant uncertainty and the performance weighting function in the frequency domain as a basis to design robust controllers with finite impulse response Q parameters. The optimal controller is given in the theorem in the form of the frequency response of a Q parameter of an internal model control structure. The design and tuning of the robust controller relies on an approximate solution to a frequency response fitting problem solved over the class of discrete-time finite impulse response Q parameters via the solution of a linear matrix inequality problem.

Keywords: Robust performance; Q-parameterization; μ synthesis; Robust control; Linear matrix inequalities; Finite impulse response

1 Introduction

For a linear time-invariant plant, model uncertainty can often be characterized as an additive unknown stable perturbation, bounded in magnitude by a given weighting function in the frequency domain. The following controller design question arises from robustness considerations: Given such a uncertain plant model, what is the best possible controller design that will optimize some performance criterion for the worst-case model in the family of perturbed plant models?

One of the main goals of feedback control is to maintain system performance despite the presence of (possibly time-varying) plant uncertainty. The technique of μ-synthesis implemented with a D-K iteration is a systematic and often successful approach for achieving this objective. But there are known problems with the D-K iteration procedure. The most significant problem is that it does not always converge to a global minimum. Even if both the K-step and D-step are convex by themselves, they are not jointly convex. In principle, the scaled H_∞-norm should decrease at each step with reasonable computational complexity, but in practice the effectiveness of D-K iteration depends on the quality of fit of the D-scales. There is a tradeoff between fitting quality and order of the D-scales. With a poor fit, the scaled H_∞-norm of the augmented plant can even increase in subsequent iterations. So, for an on-line controller redesign scheme, the computational cost of D-K iteration may be too high. Furthermore, the execution of the algorithm often needs to be overseen by the engineer to steer it out of numerical pitfalls.

Internal model control (IMC) [2] is an attractive control structure in that it allows some tuning of the parameter Q while keeping the nominal closed loop stable. IMC is based on the so-called model reference transformation, or Q-parameterization, of all stabilizing controllers for stable plants that has been widely used in the theory of optimal control, e.g., [7], [8]:

$$K = Q(I - QP)^{-1}, \qquad (1)$$

where $Q \in RH_\infty$ and P is the nominal stable plant model. We use an IMC representation of the classical feedback control loop to analyze the tradeoff between performance and robustness in single-input single-output linear time-invariant stable plants, which represents a fundamental aspect of robust control design. We present a theorem giving an explicit function of frequency that clearly exposes such a tradeoff between the maximum magnitude of the additive plant perturbation and the performance weighting function. The structured singular value is used to obtain the result of this two-disc optimization problem, where the weighting function on the control signal plays the role of a hard constraint. Furthermore, the optimal controller is given in the theorem in the form of the frequency response of the Q parameter. An application of this result to robust control is discussed.

The design and tuning of the robust tunable controller relies on an approximate solution to the frequency response fitting problem solved over the class of discrete-time, finite impulse response Q parameters via the solution of a linear matrix inequality problem. In contrast to D-K iteration, this strategy gives the designer a direct handle on the on-line tuning of the controller. This technique could support automatic or operator-initiated on-line controller tuning in response to changes in the size of the plant uncertainty or new performance requirements for the plant. Our experience with the proposed algorithm is that it has a reasonable computational cost and fast convergence. A numerical example for a second-order plant model is presented in Section 3

2 Fundamental Tradeoff Between Performance and Robustness

2.1 Problem Formulation

The conventional setup of a unity-feedback control system is shown in Figure 1.

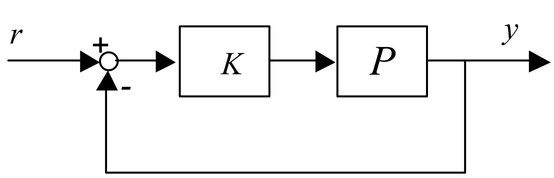

Fig.1: Unity feedback control system

Figure 2 shows a system that, through the IMC structure, is equivalent to the system in Figure 1. The filter $Q(s)$ can be designed according to the optimal procedures outlined in [2]. But here, our focus is on a setup with additive plant uncertainty, in which the performance specification and the constraint on the control signal are given by weighting functions.

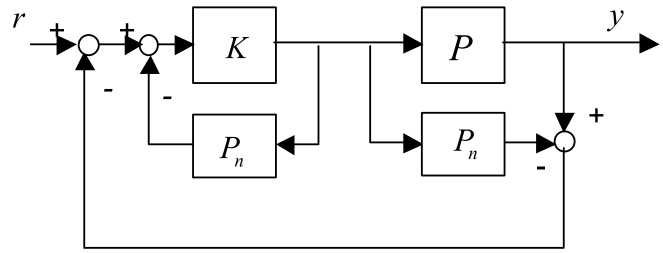

Fig.2: Equivalent IMC block diagram

Let the perturbed plant be given by:

$$P := P_n + W_a \Delta_a \quad (2)$$

where P_n is the nominal plant model, $\Delta_a \in H_\infty$ is the normalized additive perturbation with $\| \Delta_a \| < 1$, $W_a \in RH_\infty$ is the scalar weighting function characterizing the maximum size of the perturbation at each frequency. Let $\Delta := W_a \Delta_a$. The block diagram of Figure 2 boils down to the simple feedback interconnection of the Q parameter with the plant perturbation, as shown in Figure 3.

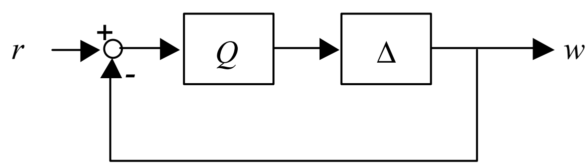

Fig.3: IMC Q parameter as a feedback around the perturbation

Clearly, the robust stability of the system in Figure 3 is determined by the small-gain theorem. Namely, the closed-loop system is well-posed and internally stable for all $\Delta \in H_\infty$, with $\|\Delta(j\omega)\| \leq |W_a(j\omega)|$ iff $\|Q(j\omega)\| < |W_a(j\omega)|^{-1}$. Therefore, the inverse of the size of the uncertainty yields a direct frequency-by-frequency constraint on the magnitude of the stable Q parameter to preserve robust stability. Based on this simple analysis, we next address the problem of finding constraints on the Q parameter to maintain robust performance.

Consider the IMC structure in Figure 4. It is well known that for a stable nominal plant, nominal closed-loop stability is guaranteed iff Q is stable. A performance weight W_p is added on the error signal for sensitivity minimization, and a weight W_u is placed on the control signal to satisfy actuator constraints. The normalized additive perturbation is pulled out and the system is rearranged into a $G-\Delta_a$ linear fractional transformation (LFT) form, as shown in Figure 5(a), where G is the closed-loop transfer matrix (which depends on Q) mapping the output of the perturbation and the reference to the input of the perturbation and the weighted error.

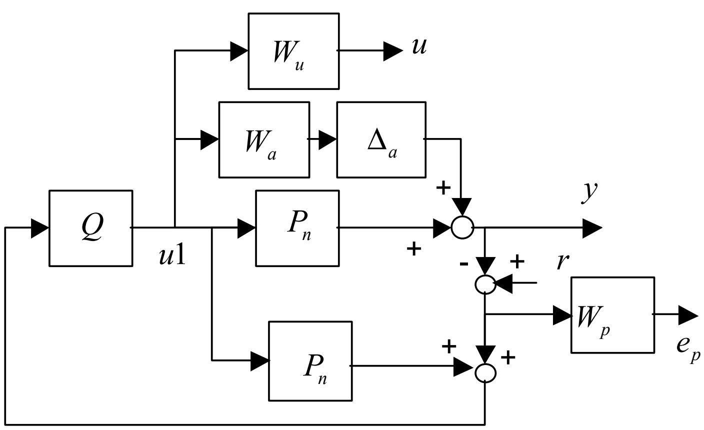

Fig.4: IMC block diagram with weighting functions

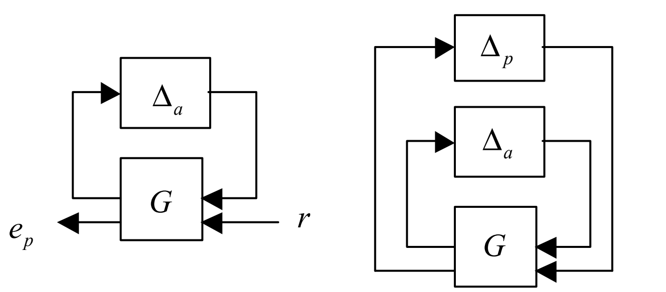

Fig.5: (a) LFT form of system; (b) Setup for μ-analysis

Robust performance of the system requires that the upper linear fractional transformation $\mathcal{F}_u\{G,\Delta_a\} = r \mapsto e_p$ satisfy $\left\|\mathcal{F}_u\{G,\Delta_a\}\right\|_\infty \le 1$, $\forall \Delta_a \in H_\infty, \left\|\Delta_a\right\|_\infty < 1$, see [1]. From the Main Loop Theorem, which states that $\left\|\mathcal{F}_u\{G,\Delta_a\}\right\|_\infty \le \mu_\Gamma[G(j\omega)]$ and the system is internally stable $\forall \Delta_a \in H_\infty, \left\|\Delta_a\right\|_\infty < \left[\mu_\Gamma(G(j\omega))\right]^{-1}$, this condition can be tested by computing the structured singular value $\mu_\Gamma[G(j\omega)]$ at all frequencies, where $\Gamma := \{diag\{\Delta_a,\Delta_p\} : \Delta_a \in \mathbb{C}, \Delta_p \in \mathbb{C}\}$ is the uncertainty structure. In [6], a sufficient condition for robust performance is provided by the following upper bound which depends explicitly on the Q parameter:

$$\mu_\Gamma\left(G\right) \le \left\|W_a Q\right\| + \left\|W_p(I-QP)\right\| \le 1, \ \forall \omega . \quad (3)$$

This sufficient condition is actually also necessary for an SISO plant, i.e., the upper bound on $\mu_\Gamma[G(j\omega)]$ is tight [1]. But first suppose the plant is square, $n \times n$. To optimize the robust performance level, a minimization problem is set up frequency by frequency:

$$\min_{\|Q\|<|W_a|^{-1},\, Q\in\mathbb{C}^{n\times n}} \left\|W_a Q\right\| + \left\|W_p(I-QP)\right\|, \quad (4)$$

where all transfer functions are evaluated at $s = j\omega$ and the constraint $\|Q\| < |W_a|^{-1}$ enforces robust stability, at the possible expense of the robust performance level as expressed by the objective function in (4). Note that the Main Loop Theorem [1] does not guarantee robust stability for all possible plant perturbations in the ball $\left\|\Delta_a\right\|_\infty < 1$ whenever $\mu_\Gamma[G] > 1$ at some frequency. However, we would like to keep robust stability as a hard constraint which is required in most applications. The optimization problem (4) is a version of the so-called two-disc optimization problem in H_∞ control theory.

2.2 Fundamental Tradeoff Between Performance and Robustness for SISO plants

For SISO plants (we use a lowercase notation for SISO transfer functions), the sufficient condition for robust performance in (3) is also necessary as the bound is tight [1]:

$$\mu_\Gamma\left(G\right) = \left|w_a q\right| + \left|w_p(1-qp)\right| \le 1, \ \forall \omega . \quad (5)$$

The minimization of $\mu_\Gamma\left(G\right)$ was addressed by the authors in [6]. Theorem 2 in [6] solves the robust performance problem with actuator constraint in that it provides a frequency-by-frequency characterization of an optimal Q parameter. Let $q_1 := qp$. The theorem is restated here for convenience.

Theorem 1

Let

$$\nu(G) := \min_{\substack{q_1 \in \mathbb{C}, \\ |q_1| < \frac{|p|}{|w_u| + |w_a|}}} \mu_\Gamma(G)$$

$$= \min_{\substack{q_1 \in \mathbb{C}, \\ |q_1| < \frac{|p|}{|w_u| + |w_a|}}} \left(\left| \frac{w_a}{p} \right| |q_1| + |w_p| |1 - q_1| \right)$$

and $q_{1opt} := \arg \min_{\substack{q_1 \in \mathbb{C}, \\ |q_1| < \frac{|p|}{|w_u| + |w_a|}}} \mu_\Gamma(G)$.

(a) Suppose $|w_p| > \left| \frac{w_a}{p} \right|$, i.e., the performance weight is larger than the relative uncertainty. Then,

$q_{1opt} = \min \left\{ 1, \frac{|p|}{|w_u| + |w_a|} \right\}$ and

$$\nu(G) = \begin{cases} \left| \dfrac{w_a}{p} \right|, & q_{1opt} = 1 \\ \dfrac{|w_a|}{|w_u| + |w_a|} + |w_p| \left(1 - \dfrac{|p|}{|w_u| + |w_a|} \right), & q_{1opt} = \dfrac{|p|}{|w_u| + |w_a|} \end{cases}$$

(b) Suppose $|w_p| \le \left| \frac{w_a}{p} \right|$, i.e., the performance weight is smaller than the relative uncertainty. Then, $q_{1opt} = 0$ and $\nu(G) = |w_p|$.

Finally $q_{opt} = q_{1opt} p^{-1}$ is the optimal frequency response of the Q parameter that minimizes $\mu_\Gamma[G]$ while satisfying the actuator constraint.

Remarks

1- The problem of finding a (sub)optimal controller that minimizes $\mu_\Gamma[G]$ directly, i.e., μ-synthesis, is a problem setup slightly different from ours. The μ synthesis approach underlines various algorithms to solve the problem such as the D-K iteration, and the Matlab μ-Analysis and Synthesis toolbox is a powerful software to implement them. In a typical μ-synthesis the minimization is performed not only on the closed-loop sensitivity, but also on the closed-loop transfer functions linking the output of the additive perturbation to its input, and the output of the fictitious perturbation associated to W_u to the reference input. On the other hand, Theorem 1 solves a minimization of $\mu_\Gamma[G]$ in which W_u plays the role of a hard constraint, as it should, instead of being part of the objective function to be minimized.

2- Note that an infinite gain controller may be obtained in some frequency bands, namely at frequencies where $q_{1opt} = \min \left\{ 1, \frac{|p|}{|w_u| + |w_a|} \right\} = 1$ in the passband on the control system. The theorem shows that robustness does not suffer in this case, so it is up to the designer to decide how high a gain is acceptable for the controller $k = \frac{q_1}{1 - q_1} p^{-1}$ and compute a corresponding upper limit on the value of q_1.

3 An LMI Approach to Designing Robust Controllers with FIR *Q* Parameters

Linear Matrix Inequalities and associated LMI techniques have emerged as powerful design tools in areas ranging from control engineering to system identification and structural design [5]. Three factors make LMI techniques appealing:

- a variety of design specifications and constraints can be expressed as LMIs,
- once formulated in terms of LMIs, a problem can be solved exactly by efficient convex optimization algorithms (the 'LMI solvers'),
- while most problems with multiple constraints or objectives lack analytical solutions in terms of matrix equations, they often remain tractable in the LMI framework. This makes LMI-based design a valuable alternative to classical 'analytical' methods.

Theorem 1 provides a characterization of the frequency response of the optimal parameter q_{opt}. The implementation procedure is to find a stable transfer function $q_{dsn}(s)$ that fits the optimal frequency response. A frequency-by-frequency optimization problem can thus be set up as follows:

$$\min_{q_{dsn} \in H_\infty} \left\| q_{dsn}(j\omega) - q_{opt}(\omega) \right\|. \qquad (6)$$

Since we have the magnitude and phase of q_{opt} at each frequency point, we can treat this optimal magnitude and phase pair as frequency-response data points. Thus, along the frequency grid, the optimization problem (6) can be solved as a matrix norm inequality problem. Next, we set up an equivalent LMI problem to get the desired Q parameter as a discrete-time system.

3.1 Setup of the LMI Problem

Given the nominal plant model and the three weighting functions, $q_{opt}(\omega)$ is first computed using Theorem 1 over a chosen frequency grid. For each frequency $\omega_k, k=1,2,...,N$ the optimization problem (6) becomes:

$$\min_{q_{dsn}\in H_\infty} \rho, \text{ such that}$$

$$\begin{bmatrix} \rho^2 & \left[q_{dsn}(j\omega_k)-q_{opt}(\omega_k)\right]^* \\ \left[q_{dsn}(j\omega_k)-q_{opt}(\omega_k)\right] & 1 \end{bmatrix} > 0 \quad (7)$$

Following [3], we restrict q_{dsn} to the space $\mathcal{S}$ of L[th]-order discrete-time, causal, real finite impulse response filters. In order to do this, we use a sampling period T_s to map the continuous-time frequency points to discrete-time frequency points over the range $\Omega_k \in [0,\pi]$ Ωi∈[0,π] using the bilinear transformation. We get

$$q_{dsn}(\Omega_k) = \sum_{m=0}^{L} q_m e^{-j\Omega_k m}, \; k=1,\ldots,N. \quad (8)$$

Note that one could use an FIR filter length of $2N$ and directly get its coefficients from the set of $2N$ real linear equations obtained by evaluating polynomial $q_{dsn}(z^{-1})$ at the N frequencies. However, although this technique provides perfect interpolation, and notwithstanding the increased filter order, it has the tendency of introducing significant oscillations in the frequency response of the filter. Thus a lower order filter is desirable. Then, the optimization problem (7) is transformed into the following convex LMI problem with complex matrices:

$$\min_{q_{dsn}\in \mathcal{S}} \rho^2, \text{ such that}$$

$$\begin{bmatrix} \rho^2 & [q_{dsn}(j\Omega_k)-q_{opt}(\Omega_k)]^* \\ [q_{dsn}(j\Omega_k)-q_{opt}(\Omega_k)] & 1 \end{bmatrix} > 0$$

$$k=1,2,...,N \quad (9)$$

where $q_{dsn}(j\Omega_k) = \sum_{m=0}^{L} q_m e^{-j\Omega_k m}, \; k=1,\ldots,N$.

Recall that the inverse DFT yields a periodic impulse response $q_{dsn}[n]$, so we have to add this constraint to the LMI problem. Suppose $q_{dsn}[n]$ is periodic, its DFT $q_{dsn}(j\Omega_k)$ is a complex sequence: $q_{dsn}(j\Omega_k) = q_{re}(j\Omega_k) + jq_{im}(j\Omega_k)$. By the Hilbert transform relationship between the real part and imaginary part of $q_{dsn}(j\Omega_k)$ [4], we have for $k=1,2,...,N$:

$$jq_{im}(j\Omega_k) = \frac{1}{N}\sum_{m=0}^{N-1} V(k-m) q_{re}(j\Omega_m), \quad (10)$$

where

$$V(k) := \begin{cases} -j2\cot(\pi k/N), & k \text{ odd} \\ 0, & k \text{ even} \end{cases}, \quad 0 \le k \le N-1 \quad (11)$$

We add this equality constraint to the LMIP. Then the optimization problem becomes:

$$\min_{q_{dsn}\in \mathcal{S}} \rho^2, \text{ such that:}$$

$$\begin{bmatrix} \rho^2 & [q_{dsn}(j\Omega_k)-q_{opt}(\Omega_k)]^* \\ [q_{dsn}(j\Omega_k)-q_{opt}(\Omega_k)] & 1 \end{bmatrix} > 0$$

$$k=1,2,...,N.$$

$$jq_{im}(j\Omega_k) = \frac{1}{N}\left\{\sum_{m=0}^{N/2} V(k-m) q_{re}(j\Omega_m) + \sum_{m=N/2+1}^{N-1} V(k-m) q_{re}(j\Omega_{N-m})\right\}, \quad k=0,\ldots,N/2 \quad (12)$$

This procedure obtains a discrete-time FIR Q parameter, which is always BIBO stable as a discrete-time system, but we need to make sure that the original IMC structure turned into a sampled-data system is still robustly stable. If pathological sampling is avoided, an argument along the lines of [2] using a zero-order hold and a bilinear discretization leads to a positive answer.

3.2 Numerical Example

The approximation of $q_{opt}(\omega)$ by $q_{dsn}(j\omega)$ is not perfect. To analyze the effect of this approximation error, we give a result to compute $\mu_\Gamma(G_{dsn})$ which turns out to be different from the optimal $\nu(G)$, where G_{dsn} is the closed-loop transfer matrix with q_{dsn}.

Proposition 1

$$\mu_\Gamma(G_{dsn}) = \left|\frac{w_a}{p}\right| aR_0 + |w_p| \sqrt{(aR_0)^2 + 1 - 2aR_0\cos\theta_0}$$

where $R_0 := \dfrac{R_{dsn}}{R_{opt}}$, $\theta_0 := \theta_{dsn} - \theta_{opt}$, $q_{opt} = R_{opt}e^{j\theta_{opt}}$, $q_{dsn} = R_{dsn}e^{j\theta_{dsn}}$, and $a := q_{1opt} \in \mathbb{R}$.

Proof

At each frequency point, we define: $q_{dsn} = q_{1dsn}p^{-1}$.

Since $q_{opt} = q_{1opt}p^{-1}$, we have: $q_{1dsn} = \dfrac{q_{dsn}}{q_{opt}} q_{1opt}$.

Then, $q_{1dsn} = \dfrac{q_{dsn}}{q_{opt}} q_{1opt} = q_{1opt}\dfrac{R_{dsn}}{R_{opt}} e^{j(\theta_{dsn}-\theta_{opt})} = aR_0 e^{j\theta_0}$.

Since we already have the filter's frequency responses q_{opt} and q_{dsn}, we can compute:

$$\begin{aligned}\mu_\Gamma(G_{dsn}) &= |w_a q| + |w_p(1-qp)| \\ &= \left|\frac{w_a}{p}\right| |q_{1dsn}| + |w_p| |1 - q_{1dsn}| \\ &= \left|\frac{w_a}{p}\right| aR_0 + |w_p| \sqrt{(aR_0)^2 + 1 - 2aR_0\cos\theta_0}\end{aligned}$$

Remark: In a good design, R_0 should be close to one and θ_0 should be close to zero.

Consider a second-order nominal plant model with additive uncertainty: $p = \dfrac{1.5(s+0.13)}{s^2+0.4s+9}$. The sampling period is $T_s = 0.025s$, the weighting functions are $w_a = 0.01\dfrac{0.1s+1}{0.02s+1}$, $w_u = .01\dfrac{10s+1}{0.2s+1}$, $w_p = \dfrac{500}{40s+1}$, and the number of frequency points is $N = 128$.

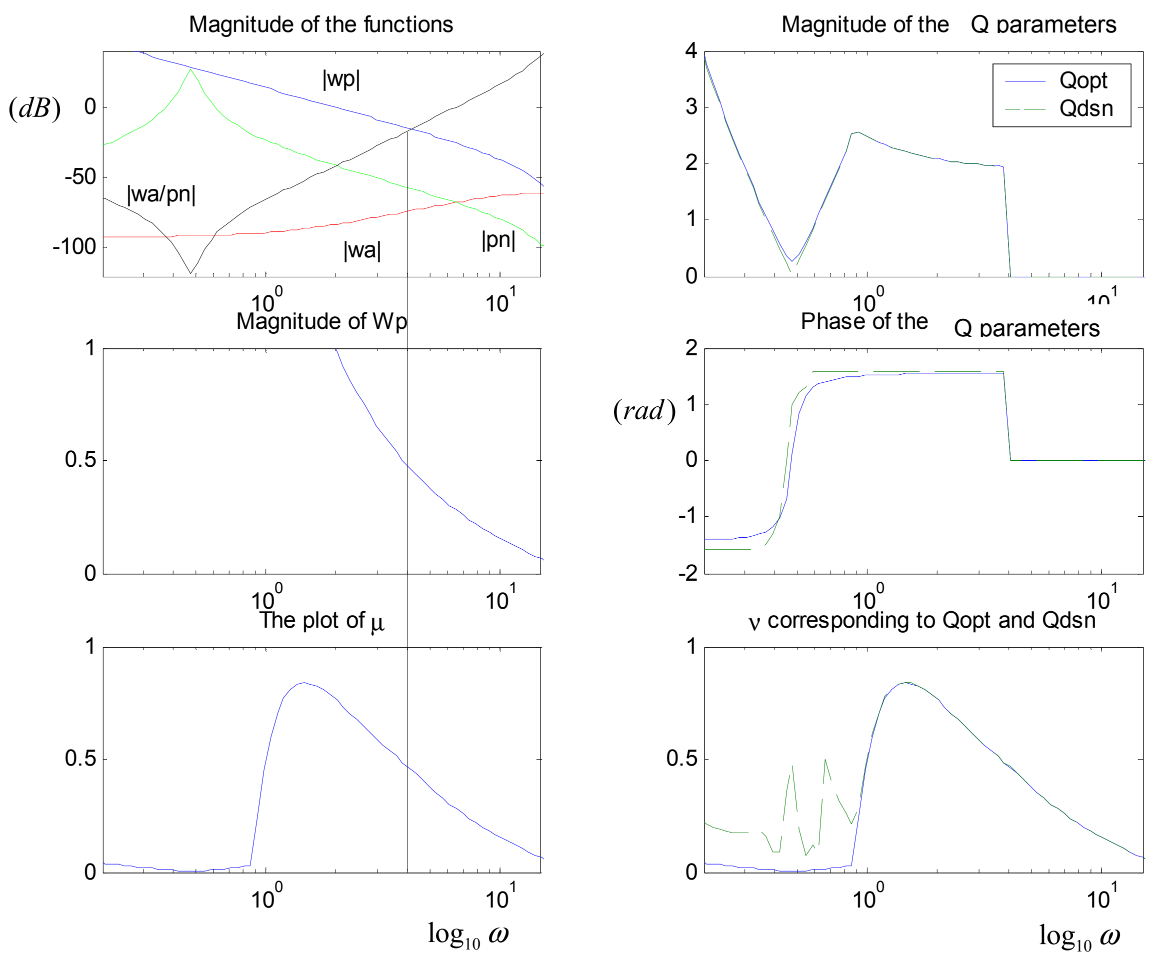

Fig.6: Optimal and achieved robust performance levels with FIR Q parameter of length $L = 128$.

The vertical line on the left of Figure 6 indicates the boundary between the two different cases in Theorem 1. Namely, below $\omega = 4.17\text{rd/s}$ we have $\left|w_p\right| > \left|\frac{w_a}{p}\right|$, and above we have $\left|w_p\right| < \left|\frac{w_a}{p}\right|$ where the optimal controller shuts off and $\nu(G) = \left|w_p\right|$. Note that all frequency responses in the figure are from discretized transfer functions. Figure 6 on the right shows the effect of the approximation error of the designed FIR parameter q_{dsn} on the optimal robust performance level $\nu(G)$ as computed by Proposition 1.

4 Conclusion

An IMC representation of the classical feedback control loop was chosen to analyze the tradeoff between performance and robustness in single-input single-output linear time-invariant stable plants, which represents a fundamental aspect of robust control design. We used a theorem from [6] giving an explicit function of frequency that clearly exposes such a tradeoff in order to obtain the optimal frequency response of the Q parameter.

The proposed design and tuning of the robust controller relies on an optimization of the optimal frequency response fitting problem solved over the class of discrete-time, finite impulse response Q parameters via the solution of a linear matrix inequality problem. Note that non-minimum phase nominal plants can easily be treated with the same approach since FIR Q parameters are always stable. The numerical example showed that an FIR Q parameter of length 128 with a reasonable fit on the optimal frequency response could maintain robust performance. The Q parameter could be obtained in less than a minute using an LMI solver on a PC.

As a bonus, the theorem provides a quick and simple way to compute the optimal robust performance level $\nu(G)$ that is equal to the structured singular value $\mu_\Gamma(G)$ minimized over the set of robustly stabilizing Q parameters.

Our proposed approach to robust control for SISO plants with additive uncertainty leads to the possibility for the control engineer to tune the controller on-line by directly changing either the uncertainty, the performance, or the actuator weighting function, as the need arises, which would trigger an LMI solver to compute a new Q parameter.

Future research will address the robust tunable control problem for unstable and multivariable plants with LFT uncertainty models.

Acknowledgment

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

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