Uncertainty models for modal parameters of flexible structures:
A complex-rational to real-rational controller design strategy.

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Abstract: In this paper, real parametric uncertainties in the modal damping ratios and frequencies of flexible structures are represented by complex uncertainties that can lead to robust controller designs satisfying robust performance specifications. These complex uncertainty blocks are useful in a $\mu-$synthesis controller design procedure. We propose two models for modal parameter uncertainties. The first model uses a coprime factorization representation of the perturbed plant while in the second model, a diagonal representation with complex eigenvalues has been used. The innovation in the second method proposed is the use of a complex-rational controller design strategy, which offers tight uncertainty bounds and leads to robust performance controller. The frequency-response of the complex-rational controller is then approximated by a real-rational controller achieving the robust performance specifications.

1- Introduction: Flexible structures are generally characterized by their damping ratios and the frequencies of their flexible modes. These parameters are subject to errors when they are estimated. These uncertainties are important and should be taken into account in a robust controller design. The proper capture of modal parameter uncertainties in dynamical models

of flexible structures for robust control has been the subject of ongoing efforts. Previous research [1,5] used additive or multiplicative uncertainty models to take into account the variation in the dynamics of the plant. Another way is to use certain heuristics to facilitate the representation of the parametric uncertainties in the flexible modes by a parametric model [2]. These heuristics represent approximations in the parameter variation that are not generally realistic and lead to conservative controller designs, i.e., designs that cannot provide the desired performance in the face of realistic levels of parametric uncertainty. Recently, a model to represent parametric uncertainties in the modes of a flexible structure has been discussed in [3]. Note that such models have been developed in [4] a few years ago. In the latter reference a model of dynamic uncertainty covering parametric variations in the flexible modes of a flexible structure has been developed. This dynamic uncertainty has the virtue to be non-conservative, but only when the frequencies of the flexible modes are close to each other.

In this paper, we propose to represent the variation in the (damping, frequency) pair of each flexible mode by a tight low-order dynamic uncertainty. Thus, we reduce the order complexity of the augmented plant by half, and transform the mixed real/complex robust performance $\mu-$design into an easier complex $\mu-$design representation. We use two techniques: the first is based on the coprime factorization framework [6] and the second uses a complex diagonal modal representation to model the dynamics of the flexible structure and to take into account the parametric uncertainties. The advantages of our methods consist in transforming the real modal parametric uncertainties into a smaller number of complex uncertainty blocks. These cover all (damping, frequency) variations in the flexible structures and reduce the complexity of the augmented plant.

The robust controller design we propose takes into account two aspects: the robustness to the uncertainties in the modal parameters, and the closed-loop performance specified for our model. The second uncertainty model proposed results in a complex-rational nominal plant. This led us to develop a new procedure to design a robust controller. A robust complex-rational controller is first obtained via a $\mu-$synthesis, and then a real controller keeping the robust performance level is designed to approximate the frequency response of the complex-rational controller.

2- Problem Setup:

Due to lack of exact knowledge of natural frequencies and damping ratios, the control designer needs to estimate uncertainty bounds for the modal parameters ζ_i and ω_i, $i = 1..n$. These bounds are used in the design to achieve the required robustness. Suppose that:

$\zeta_i = \zeta_{in} + \delta_{\zeta_i}$ and $\omega_i = \omega_{in} + \delta_{\omega_i}$.

Such that ξ_{in}, ω_{in} are the *ith* nominal damping ratio and frequency respectively. $\delta_{\zeta_i}, \delta_{\omega_i}$ represent the uncertainty in each parameter and are bounded in magnitude by:

$\left|\delta_{\omega_i}\right| \le t_{\omega_i}$ and $\left|\delta_{\zeta_i}\right| \le t_{\zeta_i}$.

t_{ζ_i}, t_{ω_i} are, respectively, the maximum of the admissible uncertainty on the damping ratio ζ_i and the frequency ω_i of each flexible mode. These uncertainties have to be taken into account in the design of a robust controller. The most used and efficient design to achieve robustness and performance for mixed type of uncertainty is the mixed μ-design based on the *D-G-K* iteration algorithm [7]. In our design, we considered, respectively for each flexible mode, two parameter variations δ_{ζ_i}, δ_{ω_i}. For these variations, no tight, efficient and realistic parametric model is available in the literature. Consequently, it's difficult to deal with real uncertainties representations to achieve robust performance criteria. Moreover, when the number of this type of uncertainty increases, it causes a problem of dimension complexity in the controller design procedure '*D-G-K* algorithm'. The dimensionality of the system augmented by the *D-G* scales will be very high and the controller synthesis will be difficult. This is more difficult when the dimensionality of the nominal flexible structure is high. Even the μ-synthesis procedure used in this paper, which is "μ controller based on H_2 design" from [9], reduces the dimension of the scales by half, a solution to the parametric uncertainty tightness and conservativeness has to be found. A novel way to deal with this constraint, under a new conception of complex controller design, is proposed.

a- Coprime factorization approach:

This is the uncertainty modeling technique inspired and adapted from the version presented in [4]. Consider the nominal dynamic equation of the flexible structure, of *m* inputs and *p* outputs, in modal coordinates:

$$\begin{aligned} &\ddot{\eta} + D\dot{\eta} + \Lambda\eta = Bu \\ &y = C\eta \end{aligned} \qquad (1)$$

such that : $D = diag\{2\zeta_i\omega_i, ..., 2\zeta_n\omega_n\}$

$$\Lambda = diag\{\omega_1^2, ..., \omega_n^2\}$$

$B \in \tilde{}^{\,n\times m}, C \in \tilde{}^{\,p\times n}$, $\{\omega_1, ..., \omega_n\}, \{\zeta_1, ..., \zeta_n\}$ are the frequencies and the damping ratios of the flexible modes. Taking the Laplace transform of (1), we obtain

$$\begin{aligned} &\eta(s) = \left(s^2 I + sD + \Lambda\right)^{-1} Bu(s) \\ &y(s) = C\eta(s) \end{aligned}$$

let us define: $G(s) = \left(s^2 I + sD + \Lambda\right)^{-1}$ and let $s^2 + as + b$ be Hurwitz with real zeros.

$G(s)$ can be written as : $G = \tilde{M}^{-1}\tilde{N}$ such that :

$$\tilde{M}(s) := \frac{1}{s^2 + as + b} diag\left\{s^2 + 2\zeta_1\omega_1 s + \omega_1^2, ..., s^2 + 2\zeta_n\omega_n s + \omega_n^2\right\}$$

$$\tilde{N}(s) := diag\left\{\frac{1}{s^2 + as + b}, ..., \frac{1}{s^2 + as + b}\right\}$$

$\tilde{M}$ and $\tilde{N}$ form a left coprime factorization of G in $_\infty$. The perturbed plant can be written as: $G_p = \left(\tilde{M} + \Delta\tilde{M}\right)^{-1}\left(\tilde{N} + \Delta\tilde{N}\right)$

$\Delta\tilde{M}, \Delta\tilde{N} \in$ $_\infty$ include all the parameters variations in the flexible modes.

$$\Delta\tilde{M} := diag\left\{\frac{\left\{2\zeta_{1n}\delta_{\omega_1} + 2\delta_{\zeta_1}\left(\omega_{1n} + \delta_{\omega_1}\right)\right\} s + 2\omega_{1n}\delta_{\omega_1} + \delta_{\omega_1}^2}{s^2 + as + b}, ..., \frac{\left\{2\zeta_{nn}\delta_{\omega_n} + 2\delta_{\zeta_n}\left(\omega_{nn} + \delta_{\omega_n}\right)\right\} s + 2\omega_{nn}\delta_{\omega_n} + \delta_{\omega_n}^2}{s^2 + as + b}\right\}$$

$$\Delta\tilde{N} := diag\{0, ..., 0\}$$

Each element of $\Delta\tilde{M}$, representing a family of strictly proper systems, can be bounded tightly by a strictly proper system using the bound of each parameter variation t_{ζ_i}, t_{ω_i} $i = 1..n$. This leads to:

$\Delta\tilde{M}$ can be bounded by the magnitude of the following weighting function [see example1]:

$$\Delta\tilde{M}_b := diag\left\{\frac{\left\{2\zeta_{1n}t_{\omega_1} + 2t_{\zeta_1}\left(\omega_{1n} + t_{\omega_1}\right)\right\} s + 2\omega_{1n}t_{\omega_1} + t_{\omega_1}^2}{s^2 + as + b}, ..., \frac{\left\{2\zeta_{nn}t_{\omega_n} + 2t_{\zeta_n}\left(\omega_{nn} + t_{\omega_n}\right)\right\} s + 2\omega_{nn}t_{\omega_n} + t_{\omega_n}^2}{s^2 + as + b}\right\}$$

The resulting bound is structured, representing tightly parametric structured block of uncertainty. This bound is more general than the bound obtained in [4], which had the assumption that the modal frequencies of the flexible structure were close to each other.

b- Modal coordinates approach :

Suppose that the state-space model in modal coordinates of a flexible structure, and particulary a flexible aircraft model, is given as:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where $A = diag\{A_i\} \;\; i = 1..n$

$$A_i = -\begin{pmatrix} -\zeta_i \omega_i + j\omega_i\sqrt{1-\zeta_i^2} & 0 \\ 0 & -\zeta_i \omega_i - j\omega_i\sqrt{1-\zeta_i^2} \end{pmatrix}$$

$(sI - A) = diag\{sI - A_i\}$.

we use the approximation $\zeta_i << 1$, which can be verified in many examples of flexible structures models. Let us define the complex-rational factorization in $_\infty$:

$G(s) = (sI - A)^{-1} = \tilde{M}(s)^{-1}\tilde{N}(s)$ and $\tilde{M} = diag\{\tilde{M}_i\}$:

$$\tilde{M}_i = \frac{1}{s+\alpha}(sI - A_i) = \frac{1}{s+\alpha}\begin{pmatrix} s+\omega_i(\zeta_i + j) & 0 \\ 0 & s+\omega_i(\zeta_i - j) \end{pmatrix} \; \tilde{N} = diag\left\{\frac{1}{s+\alpha},...,\frac{1}{s+\alpha}\right\}$$

where $s+\alpha$: Hurwitz with real zero.

The perturbed model can be written as:

$$G_p = \left(\tilde{M} + \Delta\tilde{M}\right)^{-1}\left(\tilde{N} + \Delta\tilde{N}\right)$$

$$= diag\left\{\begin{pmatrix} s+\omega_{in}(\zeta_{in} + j) + \omega_{in}\delta_{\xi_i} + \delta_{\omega_i}(\zeta_{in} + j) + \delta_{\omega_i}\delta_{\xi_i} & 0 \\ 0 & s+\omega_{in}(\zeta_{in} - j) + \omega_{in}\delta_{\zeta_i} + \delta_{\omega_i}(\zeta_{in} - j) + \delta_{\omega_i}\delta_{\zeta_i} \end{pmatrix}^{-1}\right\}$$

$i = 1..n$

$$\Delta\tilde{M} := \frac{1}{s+\alpha} diag\left\{\begin{pmatrix} \omega_{in}\delta_{\zeta_i} + \delta_{\omega_i}(\zeta_{in} + j) + \delta_{\omega_i}\delta_{\zeta_i} & 0 \\ 0 & \omega_{in}\delta_{\zeta_i} + \delta_{\omega_i}(\zeta_{in} - j) + \delta_{\omega_i}\delta_{\zeta_i} \end{pmatrix}\right\}$$

$i = 1..n$

$\Delta\tilde{N} := diag\{0,...,0\}$.

The magnitude of each sub-block $\Delta\tilde{M}_i$, such that $\Delta\tilde{M} = diag\{\Delta\tilde{M}_i\}$, can be tightly bounded by using the maximum on each parameter interval t_{ζ_i}, t_{ω_i}. Because we are interested by the magnitude of the uncertainty that the system is subjected to, it is possible to use the complex-rational weighting function $\Delta\tilde{M}_b$ given below to bound $\Delta\tilde{M}$ in the design [see example2].

$$\Delta\tilde{M}_b(s) := \frac{1}{s+\alpha} diag\left\{\begin{matrix} \omega_{in}t_{\zeta_i} + t_{\omega_i}\zeta_{in} + t_{\omega_i}t_{\zeta_i} + jt_{\omega_i} & 0 \\ 0 & \omega_{in}t_{\zeta_i} + t_{\omega_i}\zeta_{in} + t_{\omega_i}t_{\zeta_i} - jt_{\omega_i} \end{matrix}\right\}$$

The uncertainty block in the design will be tightly represented by a modal bound $\Delta\tilde{M}_b(s)$ repeated as many times as there are uncertain parameter pairs of flexible modes.

3- Control design

Our control design methodology is based on the coprime factorization obtained from both uncertainty representation approaches. We use the μ-design technique to take into account the robust performance specification. The design concept is explained by Figure 1:

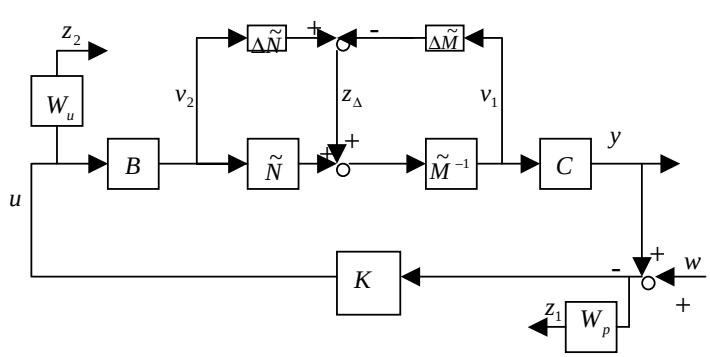

Figure 1

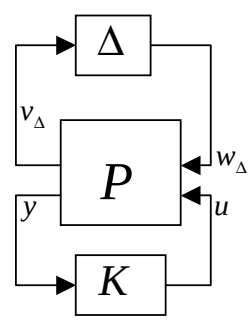

Figure 2

$\Delta\tilde{N} := diag\{0,...,0\}$.

The robust performance is taken into account in the $\mu-$design by including a fictitious uncertainty Δ_p linking the input (w) to the outputs (z_1, z_2). We transform the scheme given in the previous figure to the classical $\mu-$setup. We obtain the design given by figure2:

Where: $w_\Delta = (v_1, v_2, w)^t, (z_\Delta, z_1, z_2)$ $\Delta = \begin{bmatrix} \Delta_c & 0 \\ 0 & \Delta_p \end{bmatrix}$; $\Delta_c = \left[\Delta\tilde{M} \quad -\Delta\tilde{N}\right]$

$$P = \left[\begin{array}{c|c|c} \tilde{M}^{-1} & 0 & \tilde{N} \\ 0 & 0 & B \\ -W_p C\tilde{M}^{-1} & -W_p & -W_p \tilde{N} B \\ 0 & 0 & W_u \\ C\tilde{M}^{-1} & 0 & \tilde{N}B \end{array}\right]$$

Since in our case $\Delta\tilde{N}$ is null, then $\Delta_c = \Delta\tilde{M}$ which is in diagonal form and contains the complex uncertainties that cover the parameters variations in each flexible mode.

4- Simulation and new control design strategy :

To validate our methods, we chose two flexible systems representing the well known three-mass system [8].

Example1: In the first example, the damping ratios and the frequencies of the flexible modes are $\xi_1 = 0.072$, $\xi_2 = 0.023$, $\xi_3 = 0.016$, $\omega_1 = 0.91$, $\omega_2 = 1.81$, $\omega_3 = 1.5$. The input matrix B is given by $B = \begin{bmatrix} 0 & 2.4 & 4.4 \end{bmatrix}^T$. We suppose that the uncertainties in ξ_i, ω_i $i = 1,.3$ are 10% and 0.1% respectively. Thus, the controller to be designed has to be more robust against variations in the damping ratios. The level of performance specified is given by $W_p = \dfrac{s+3}{10.5s + 0.03}$ and the constraint on the controller is specified by : $W_u = 1/0.7$. In this example, we use the first approach described above. Figure 3 gives the Bode plots of the first diagonal entry in $\Delta\tilde{M}$ which concerns all the parameter variations in the coefficients of the first flexible mode and the magnitude of the first diagonal entry of $\Delta\tilde{M}_b$. It is easy to see that this magnitude tightly bounds all the variations in the first diagonal entry of $\Delta\tilde{M}$ **.** This observation is valid for the other flexible modes as well. Thus, the weighting function $\Delta\tilde{M}_b$ represents a tight bound of $\Delta\tilde{M}$.

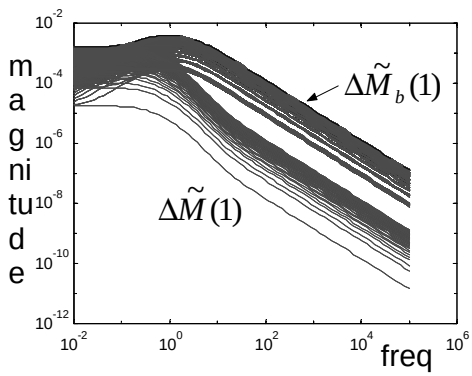

Figure 3

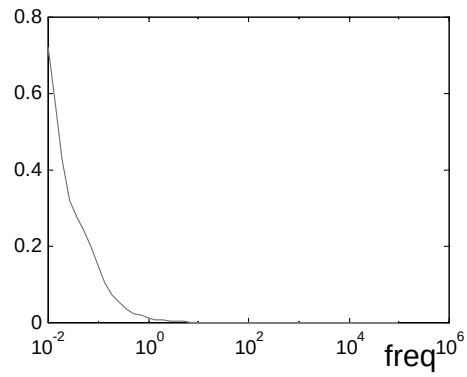

Figure 4

A μ controller is designed for this example taking into account the complex uncertainty $\Delta\tilde{M}_b$ covering all the parameters variations in all the flexible modes. The figure 4 gives the $\mu-$upper bound that we could get with this μ controller.

The $\mu-$upper bound obtained, 0.72, shows that our controller design is effectively guaranteeing the robust performance specified. It is, also, robust against the parameters uncertainties that existing in the coefficients of the flexible modes.

Example2*:* The second example representing the three mass system is given in a complex modal approach representation detailed above. The damping ratios and the frequencies of the flexible modes are $\xi_1 = 0.025$ $\xi_2 = 0.012$ $\xi_3 = 0.0043$, $\omega_1 = 5.12$ $\omega_2 = 2.43$ $\omega_3 = 0.87$ **.** The uncertainties that the damping ratios and the frequencies of the flexible modes are subject to, represent 10% and 3% respectively. These amount of uncertainties, especially those in the frequencies, are highly demanding for a robust control design.

The weighting functions W_p and W_u are are the same than in the first example.

Figure5 shows that the magnitude of the first element in $\Delta\tilde{M}_b$ is bounding tightly the all the variations in the first element of $\Delta\tilde{M}$ **.** The magnitude of both second elements is the same than in the first elements respectively. The obsevation of figure 5 stills valid for the other flexible modes. Thus $\Delta\tilde{M}_b$ is a good and tight complex uncertainty candidate bounding the paramectric variations in $\Delta\tilde{M}$.

The μ complex controller designed in this case achieves the robust performance criteria specified. Figure6 shows the $\mu-$upper bound obtained for this design. It is equal to 0.99. We have to note that the μ controller designed is not optimal, and this is because of the uncertainty block structure. This problem of optimality of the μ-design has been mentioned in the litterature [7]. However the μ-design stills the best strategy to deal concretely with the uncertainties.

Thus this complex-rational controller design garantees all specifications of performance specified. It also permitted us to deal efficiently and tightly with the parametric uncertainties under a complex representation. An amount of 3% of uncertainty in the frequecies could be tolerated. This amount of tolerence in the frequency uncertainties was rarely reached in the litterature.

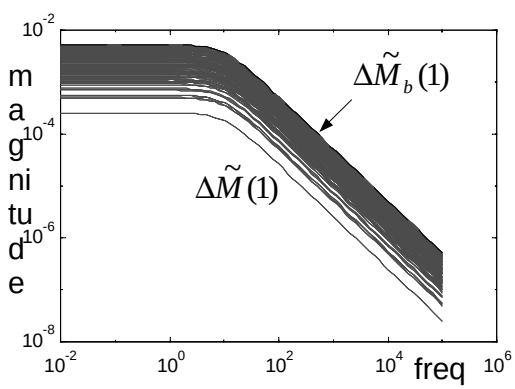

Figure 5

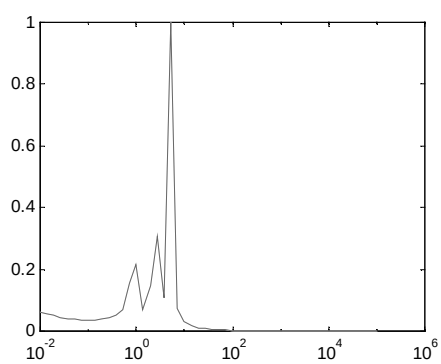

Figure 6

After designing a complex-rational controller achieving robust performance specifications and taking advantages of complex-rational controller design strategy, we have to recover a realizable real-rational controller. This controller has to maintain the performance specifications obtained by the complex-rational controller. The problem can be posed as :

$\min_{K_{rea}} \|K - K_{rea}\|_\infty$ such that :

K is the complex-rational controller designed and K_{rea} is the real controller to be found. By assuring the closness of the behavior of the two controllers using the hinfinity norm, we can be sure that both controllers will act by the same manner on the generalized plant when closing the feedback loop. Thus we will approach the level of performance obtained by the

complex-rational controller by using a real-rational controller. One way to solve the minimisation above is to fit the magnitude and the phase generated by the complex-rational controller, by a real-rational system K_{rea}. Tools, which has the ability to approach a solution for that fitting procedure, can be found in the Identification toolbox [10]. One of the benifits of this real procedure design is the reduction of the real-rational controller. In fact the constraint to maintain a minimum of the structure of the original controller while reduction procedure, is not posed here because of the diffrence in the structure of the original complex-rational controller and the real-rational controller to be found. Thus a low order, real-rational controller, achieving robust performance criteria, can be obtained.

A real-rational controller of order 2, achivieng a robust performance indice of 0.85, was generated for our example. The robust performance level for the real controller is shown by figure7. The robust complex-rational controller is of order 28.This high dimantionality is because of the number and the order of the scales in the μ-design. Due to the non-optimality of the μ-design, the robust performance of the real-rational controller was better than the one obtained by the original complex-rational controller.

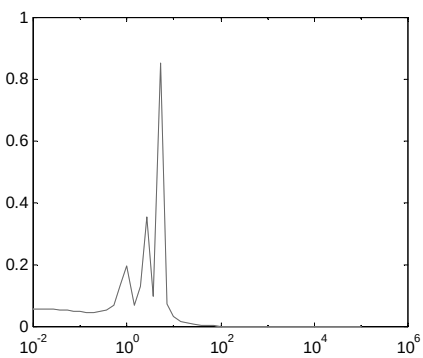

Figure 7

5- Conclusion:

In this paper we proposed models to represent parametric variations in the coefficients of the flexible modes of flexible structures. We used less conservative and tight complex-rational uncertainties for a $\mu-$setup design. We reduce at minimum by half the dimensionality of the augmented plant. We introduced a new complex-rational modal approach that leads to complex-rational design strategy rather than real. Finally a real-rational controller, keeping the performance reached by the complex modal approach design, was recovered.

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