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Consistency of Experimental Frequency-Response Data with Coprime Factor Plant Models

Benoit Boulet, Bruce A. Francis
Department of Electrical and Computer Engineering,
University of Toronto, Toronto, Ontario, Canada M5S 1A4
benoit@control.utoronto.ca francis@control.utoronto.ca

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The model/data consistency problem for coprime factorizations (CF) considered here is this: Given some experimental frequency-response data obtained on a system, show that these data are consistent with the family of perturbed factor models. In other words, does there exist a perturbation belonging to the uncertainty set such that the input-output data can be reproduced by the perturbed model? One motivation for using a coprime factorization approach is that it is well suited for modeling low-damped dynamics of large flexible space structures [1]. The model/data consistency problem boils down to the existence of an interpolating function in $\mathcal{RH}_\infty(\mathbb{D})$ which evaluates to a finite number of complex matrices of compatible dimensions at a finite number of points on the unit circle. The main result is a theorem on boundary interpolation in $\mathcal{RH}_\infty(\mathbb{D})$. This necessary and sufficient condition allows us to devise a simple test consisting of computing minimum-norm solutions to an underdetermined linear complex matrix equation to check if the perturbed factorization is consistent with the data. Left-coprime factorizations (LCF) are studied, but the results also apply to right-coprime factorizations (RCF). Due to space limitations many details which can be found in the full report [2] were left out.

Let $G(s)$ be a real-rational transfer matrix and let $\tilde{M}(s)$ and $\tilde{N}(s)$ be an LCF of $G(s)$ in $\mathcal{RH}_\infty$. Then $G(s)$ can be written as $\tilde{M}(s)^{-1}\tilde{N}(s)$. The complex argument s is dropped hereafter to ease the notation. Let the perturbed plant model G_p be expressed as a perturbed left factorization with $\tilde{M}_p, \tilde{N}_p \in \mathcal{RH}_\infty$

$$G_p = \tilde{M}_p^{-1}\tilde{N}_p \ , \qquad (1)$$

where $\tilde{M}_p = \tilde{M} + \Delta M$, $\tilde{N}_p = \tilde{N} + \Delta N$; ΔM, $\Delta N \in \mathcal{RH}_\infty$. Define the uncertainty matrix $\boldsymbol{\Delta} := [\Delta N \ \ -\Delta M]$. This matrix is defined because the result on robust stability of the system in Figure 1

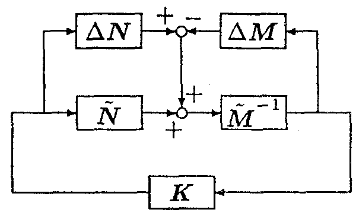

Figure 1: Feedback control of a perturbed LCF model

for a coprime factor plant description is expressed in terms of a norm (maximum singular value) bound on $\boldsymbol{\Delta}(j\omega)$ [5], [6]. Define the uncertainty set

$$\mathcal{D}_r := \{\boldsymbol{\Delta} \in \mathcal{RH}_\infty \mid \|\boldsymbol{\Delta}(j\omega)\| < |r(j\omega)|, \ \forall\omega \in \mathbb{R}\} \qquad (2)$$

and the family of plants

$$\mathcal{P} := \{G_p \mid \boldsymbol{\Delta} \in \mathcal{D}_r\} \ , \qquad (3)$$

where r is a scalar-valued function in $\mathcal{RH}_\infty$ assumed small enough so that $\tilde{M}_p$ is nonsingular. For example, we might require that $\underline{\sigma}(\tilde{M}(j\omega)) > |r(j\omega)|$ for some $\omega \in \mathbb{R}$. The exact result due to Vidyasagar [6] is

Theorem 1 *The closed-loop system of Figure 1 with controller K is internally stable for every $G_p \in \mathcal{P}$ iff (a) K internally stabilizes G, and $\forall\omega \in \mathbb{R}$,*

$$(b) \left\| \begin{bmatrix} K(I+GK)^{-1}\tilde{M}^{-1} \\ (I+GK)^{-1}\tilde{M}^{-1} \end{bmatrix} (j\omega) \right\| \leq \frac{1}{|r(j\omega)|} \ .$$

In order to be able to use this result in the design of a robust controller for a real plant, one has to construct and modify the bound $|r(j\omega)|$ until it makes sense for the uncertainty in the physical system. The consistency test proposed here is suitable for doing that.

Suppose we are given a family of plants $\mathcal{P}$. The exact statement of the model/data consistency problem considered here is the following. Suppose that a frequency-response experiment consisting of measuring perfectly the frequency response $G_r(j\omega)$ of a m-input, p-output system at $\omega = \omega_1, \ldots, \omega_N$ is performed. Could the experimental frequency-response data have been produced by at least one model in $\mathcal{P}$? Or, in other words, does there exist a fixed $\Delta \in \mathcal{D}_r$ such that the corresponding perturbed model G_p interpolates the complex matrices $G_r(j\omega_i)$ at $\omega = \omega_1, \ldots, \omega_N$?

Premultiplying (1) by $\tilde{M} + \Delta M$ and taking ΔM and ΔN onto the left-hand side yields

$$\Delta M G - \Delta N = \tilde{N} - \tilde{M} G \,. \qquad (4)$$

Let $U := \tilde{N} - \tilde{M}G$, $W := \begin{bmatrix} -I \\ -G \end{bmatrix}$ and $s = j\omega$. Then (4) can be written as

$$\Delta(j\omega) W(j\omega) = U(j\omega) \,, \qquad (5)$$

where $W(j\omega) \in \mathbb{C}^{(m+p)\times m}$, $U(j\omega) \in \mathbb{C}^{p\times m}$. Equation (5) is just an underdetermined system of linear equations over the field $\mathbb{C}$.

It is assumed that the measurements are perfect. Let $G_i := G_r(\omega_i)$ for $i = 1, \ldots, N$, with similar definitions for Δ_i, W_i and U_i. We seek a test that would show whether or not there exists a rational matrix Δ that belongs to the uncertainty set $\mathcal{D}_r$ and satisfies the interpolation constraints given by (5) at the frequencies $\{\omega_1, \ldots, \omega_N\}$. This is done in two steps: First, solve the matrix equation (5) with $G = G_i$ for Δ_i, $i = 1, \ldots, N$, such that Δ_i has minimum norm. Note that the matrix W_i has full column rank, and a minimum-norm solution $\hat{\Delta}_i$ to (5) is given by the premultiplication of the left pseudoinverse of W_i by U_i:

$$\hat{\Delta}_i = U_i (W_i^* W_i)^{-1} W_i^* \,. \qquad (6)$$

If $\|\hat{\Delta}_i\| \geq |r(j\omega_i)|$ for any $i \in \{1, \ldots, N\}$, then the test fails: The family of coprime factorizations cannot account for the frequency-response data. If $\|\hat{\Delta}_i\| < |r(j\omega_i)|$, $\forall i \in \{1, \ldots, N\}$, then we must show that there exists a matrix-valued function $\Delta \in \mathcal{RH}_\infty$ taking on the complex matrix values $\{\hat{\Delta}_i\}_{i=1}^N$ at the frequencies $\{\omega_i\}_{i=1}^N$ and such that $\|\Delta(j\omega)\| < |r(j\omega)|$, $\forall \omega \in \mathbb{R}$. This is the second step in the test. Now the problem can be scaled as follows: Find a matrix $\Phi \in \mathcal{RH}_\infty^{p\times(m+p)}$ interpolating the product $r^{-1}(j\omega_i)\hat{\Delta}_i$ at $s = j\omega_i$, $i = 1, \ldots, N$ and such that $\|\Phi\|_\infty < 1$. The interpolation problem in $\mathcal{RH}_\infty$ of the right half-plane is then transformed to an interpolation problem in $\mathcal{RH}_\infty(\mathbb{D})$ by using the scalar bilinear transformation $s \mapsto z$ defined by $z = (1-s)/(1+s)$ which maps the closed right half-plane onto the closed unit disk $\overline{\mathbb{D}}$. Thus the boundary interpolation problem can be stated as

Problem 1 *Given a set of distinct points $\{e^{j\theta_i}\}_{i=1}^r$ on the unit circle $\partial\mathbb{D}$ and a set $\{\tilde{\Phi}_i\}_{i=1}^r$ in $\mathbb{C}^{m\times n}$ satisfying $\|\tilde{\Phi}_i\| < 1$, does there exist a function $\tilde{\Phi} \in \mathcal{RH}_\infty(\mathbb{D})$, $\|\tilde{\Phi}\|_\infty < 1$ such that $\tilde{\Phi}(e^{j\theta_i}) = \tilde{\Phi}_i$, $i = 1, \ldots, r$?*

The main result on boundary interpolation for non-square matrices proved in [2] gives an answer to Problem 1:

Theorem 2 *Given a set of distinct points $\{e^{j\theta_i}\}_{i=1}^r$ on $\partial\mathbb{D}$ and a set $\{\tilde{\Phi}_i\}_{i=1}^r$ in $\mathbb{C}^{m\times n}$, there exists a function $\tilde{\Phi} \in \mathcal{RH}_\infty(\mathbb{D})^{m\times n}$ satisfying $\tilde{\Phi}(e^{j\theta_i}) = \tilde{\Phi}_i$, $i = 1, \ldots, r$, $\|\tilde{\Phi}\|_\infty < 1$ iff $\|\tilde{\Phi}_i\| < 1$, $\forall i \in \{1, \ldots, r\}$.*

The proof uses the results in [4] and [3] on the matrix Nevanlinna-Pick problem.

With Theorem 2 in hand, checking consistency of the perturbed CF model with the experimental frequency-response data becomes a simple matter. It suffices to compute the complex matrices $\hat{\Delta}_i$ for $i = 1, \ldots, N$ and to check that $\|\hat{\Delta}_i\| < |r(j\omega_i)|$, $\forall i = 1, \ldots, N$. If this inequality does not hold for some $j \in \{1, \ldots, N\}$, then no perturbation of the coprime factors in $\mathcal{D}_r$ could have produced the data. However, the bound $|r(j\omega)|$ can be modified such that the inequality above is satisfied for all i. One can now see how r can be constructed and improved as new experimental data become available. If it is suspected that the frequency-response data are noisy, a more conservative r may be chosen.

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