Delay-dependent finite-horizon time-varying bounded real lemma for uncertain linear neutral systems

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Abstract

A bounded real lemma (BRL) is presented here for continuous-time, finite-horizon, time-varying linear neutral systems with parametric uncertainties entering all the matrices of the system representation. The lemma uses a solution of a set of matrix inequalities by employing a descriptor model transformation of the system. The conditions which guarantee disturbance attenuation are dependent on the value of the time-delay and its rate of change. In the infinite-horizon case, the solution of the robust H_∞ conrtrol problem is obtained in terms of linear matrix inequalities (LMI). A numerical example is presented and illustrate the effectiveness of the proposed approach.

1. Introduction

In recent years, much work has been devoted to the analysis and synthesis of controllers for state-delayed systems with or without parametric uncertainties; see [1] for earlier work in this area. This interest is strongly motivated by the fact that delays and uncertainties are the two most important causes of instability. Furthermore, both delays and uncertainties occur frequently in the chemical and process industries, which provides another reason for the study of new and less conservative stability conditions.

The H_∞ design problem for the general case of neutral (descriptor) systems , in which the time-delays can appear both in the state and its derivative, has so far obtained relatively little attention [2]. Unlike simple retarded systems, neutral systems are particularly sensitive to delays and can be easily destabilized [3-4].

Most H_∞ design approaches for retarded systems refer to the infinite-horizon case while, to our knowledge, the more challenging finite-horizon time-varying case has only been discussed in [5].

In this context, the contribution of this paper is as follows:

- The finite-horizon time-varying robust H_∞ control problem is addressed here for linear neutral systems in full generality, with norm-bounded parametric uncertainties entering all the matrices in the system representation.
- The least conservative approach to the derivation of the bounded real lemma, as proposed in [6] for the infinite-horizon problem (without parametric uncertainties), is used in this paper. However, the approach of [6] is modified and adapted here to suit the finite-horizon time-varying case with parametric uncertainties.
- The value of the time-delay as well as its rate of change are taken into account in the design method presented , which further reduces conservativeness of the approach.
- When solved numerically, the solution to the finite-horizon time-varying case is shown to result in a set of linear matrix inequalities (LMI) at each time step. The latter can easily be solved using recently developed algorithms [7].
- The solution to the infinite horizon case which incorporates all the parametric uncertainties is also presented and leads to a design in terms of a set of linear matrix inequalities.

2. Adopted notation

The following notation is adopted. For any matrix A, the expression A^T denotes its transpose. The notation $diag\{A\}$ and $col\{v_1, v_2, ..., v_n\}$ is used for the diagonal of a matrix A and for a column vector formed by stacking column vectors $v_1,\ v_2, ...v_n$. As usual, R^n is the n dimensional Euclidean space with vector norm $\|\ .\ \|$, $R^{n\times m}$ is the set of all $n\times m$ real matrices, and for any matrix $P\in R^{n\times n}$, the inequality $P>0$, signifies that P is symmetric and positive definite. The standard notation of $L_2^q[a,b]$ is adopted for the space of all functions $f: R\to R^q$ which are Lebesque integrable in the square over the interval $[a,b]$, with the standard norm $\|\ .\ \|_{L_2}$. The symbol $C_n[a,b]$ is used to denote the space of all continuous functions ϕ: $[a,b]\to R^n$, with the standard supremum norm $\|\ .\ \|_\infty$. Denote $x_t(\theta)=x(t+\theta)$,$\theta\in[-h,0]$.

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3. Problem statement

Consider the following system:

$$\dot{x}(t)-\bar{G}(t)\dot{x}(t-g(t))=\bar{A}(t)x(t)+\bar{A}_1(t)x(t-h(t)) + B_1(t)w(t) \quad (1)$$

$$x(t)=0, \qquad \forall t\le 0$$

$$z(t)=col\{\bar{C}_0(t)x(t)+D(t)w(t),\bar{C}_1(t-h(t))x(t-h(t)) ,\bar{C}_2(t-g(t))x(t-g(t))\} \quad (2)$$

with, $\bar{G}(t)=G(t)+\Delta G(t)$, $\bar{A}(t)=A(t)+\Delta A(t)$,

$$\bar{A}_1(t)=A_1(t)+\Delta A_1(t) \quad (3)$$

$$\bar{C}_0(t)=C_0(t)+\Delta C_0(t), \qquad \bar{C}_1(t)=C_1(t)+\Delta C_1(t),$$

$$\bar{C}_2(t)=C_2(t)+\Delta C_2(t) \quad (4)$$

where $x(t)\in R^n$ is the system state vector, $w\in L_2^q[0,T]$ is the exogenous disturbance signal with $T\in R^+$ denoting the control time horizon, and $z(t)\in R^p$ is the output to be attenuated. The system delays $h(t)>0$ and $g(t)>0$ are assumed to be some unknown functions of time. The matrices $G(t)$, $A(t)$, $A_1(t)$, $B_1(t)$, $C_0(t)$, $C_1(t)$, $C_2(t)$, and $D(t)$ are bounded, real, time-varying, with piece-wise continuous entries over $[0,T]$, and are assumed to be known exactly. The matrices $\Delta G(t)$, $\Delta A(t)$, $\Delta A_1(t)$, $\Delta C_0(t)$, $\Delta C_1(t)$ and $\Delta C_2(t)$ are real-valued, represent the norm-bounded parameter uncertainties, and are assumed to be of the following form:

$$\Delta G(t)=L_g(t)F_g(t)E_g(t), \qquad \Delta A(t)=L(t)F(t)E(t),$$

$$\Delta A_1(t)=L_1(t)F_1(t)E_1(t) \quad (5)$$

$$\Delta C_0(t)=N_0(t)F_c(t)E_c(t), \qquad \Delta C_1(t)=N_1(t)F_c(t)E_c(t),$$

$$\Delta C_2(t)=N_2(t)F_c(t)E_c(t) \quad (6)$$

where $F_g(t)\in R^{i_g\times j_g}$, $F(t)\in R^{i\times j}$, $F_1(t)\in R^{i_1\times j_1}$ and $F_c(t)\in R^{i_c\times j_c}$ are real, uncertain, time-varying matrices with Lebesgue measurable entries which, additionally, meet the requirements that : $F_g(t)F_g^T(t)\le I$, $F(t)F^T(t)\le I$, $F_1(t)F_1^T(t)\le I$ and $F_c(t)F_c^T(t)\le I$. The matrices $L_g(t)$, $L(t)$, $L_1(t)$, $N_0(t)$, $N_1(t)$, $N_2(t)$, $E_g(t)$, $E(t)$, $E_1(t)$ and $E_c(t)$ are known, real, time-varying, with piece-wise continuous entries over $[0,T]$, and characterize the way in which the uncertain parameters of $F_g(t)$, $F(t)$, $F_1(t)$ and $F_c(t)$ enter the nominal matrices $G(t)$, $A(t)$, $A_1(t)$, $C_0(t)$, $C_1(t)$ and $C_2(t)$.

The delays h and g in the system are functions of time and are assumed to be continuously differentiable, with their respective amplitudes and rates of change bounded as follows:

$$0\le h(t)\le h_{\max}, \qquad 0\le g(t)<\infty, \qquad \text{for } t\in[0,T] \quad (7)$$

$$0\le \dot{h}(t)\le \alpha<1, \qquad 0\le \dot{g}(t)\le \beta<1, \qquad \text{for } t\in[0,T] \quad (8)$$

where $h_{\max}$, α and β are given positive constants.

Also, $\bar{A}_1(t)$ is bounded as follows (see Remark 2 in Appendix):

$$\bar{A}_1^T(t)\bar{A}_1(t)\le \bar{A}_{1,\max}^T(t)\bar{A}_{1,\max}(t), \quad \text{for all } t\in[0,T] \quad (9)$$

where $\bar{A}_{1,\max}(t)$ is a continuously differentiable given matrix for all $t\in[0,T]$.

The robust H_∞ disturbance attenuation problem (RDAP):

For any scalar $\gamma>0$, let the following performance index be defined:

$$J=x^T(T)P_{1T}x(T)+\int_0^T\left[z^T(t)z(t)-\gamma^2w^T(t)w(t)\right]dt \quad (10)$$

where P_{1T} is a given weighting matrix for the terminal state $x(T)$.

The design problem with disturbance attenuation level γ can now be translated into finding conditions for system *(1)* and *(2)* which yield $J<0$, for all disturbances $w\in L_2^q[0,T]$, subject to the usual assumption that $x(t)=0$, for all $t\le 0$, and $w(t)=0$, for all $t<0$.

4. Main results

The following theorem delivers the main result of this section; for simplicity of notation the time parameter t is omitted in the entries of all matrices.

Theorem 1. (Bounded Real Lemma : finite-horizon case) *Consider the system (1)-(2). For a given $\gamma>0$ and a given symmetric, positive-definite matrix P_{1T}, suppose that there exist $n\times n$-matrices: $P_1(t)>0$ such that $P_1(T)=P_{1T}$, $P_2(t)$, $P_3(t)$, $S(t)=S^T(t)>0$ with $\dot{S}(t)\le 0$, $U(t)=U^T(t)>0$ with $\dot{U}(t)\le 0$, $W_1(t)$, $W_2(t)$, $W_3(t)$, $W_4(t)$, and $R_1(t)=R_1^T(t)$, $R_2(t)$, $R_3(t)=R_3^T(t)>0$, $t\in[0,T]$, and positive scalars $\varepsilon_i(t)$; $i=1...8$, which satisfy the following matrix inequality:*

$$\begin{bmatrix}\Omega_1 & \Omega_2\\ \Omega_2^T & \Omega_3\end{bmatrix}<0 \quad (11)$$

with,

$$\Omega_1=\begin{bmatrix} \Psi_1 & \Psi_2 & P_2^TB_1 & h_{\max}\Phi_1 & -W_3^TA_1-\varepsilon_4E_1^TE_1 & P_2^TG & \tilde{C}^T & C_0^T \\ \Psi_2^T & \Psi_3 & P_3^TB_1 & h_{\max}\Phi_2 & -W_4^TA_1 & P_3^TG & 0 & 0 \\ B_1^TP_2 & B_1^TP_3 & -\gamma^2I & 0 & 0 & 0 & 0 & D^T \\ h_{\max}\Phi_1^T & h_{\max}\Phi_2^T & 0 & -h_{\max}R & 0 & 0 & 0 & 0 \\ -A_1^TW_3-\varepsilon_4E_1^TE_1 & -A_1^TW_4 & 0 & 0 & -(1-\alpha)S+\varepsilon_4E_1^TE_1 & 0 & 0 & 0 \\ G^TP_2 & G^TP_3 & 0 & 0 & 0 & -(1-\beta)U+\varepsilon_3E_g^TE_g & 0 & 0 \\ \tilde{C} & 0 & 0 & 0 & 0 & 0 & -I & 0 \\ C_0 & 0 & D & 0 & 0 & 0 & 0 & -I_p \end{bmatrix} \quad (12)$$

$$\Omega_2=\begin{bmatrix} P_2^TL & P_2^TL_1 & P_2^TL_g & W_3^TL_1 & 0 & C_0^TN_0 & C_1^TN_1 & C_2^TN_2 \\ P_3^TL & P_3^TL_1 & P_3^TL_g & W_4^TL_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D^TN_0 & 0 & 0 & 0 \\ \multicolumn{8}{c}{0} \end{bmatrix} \quad (13)$$

$$\Omega_3=diag\left\{-\varepsilon_1I,-\varepsilon_2I,-\varepsilon_3I,-\varepsilon_4I,-\varepsilon_5I,-\left(\varepsilon_6I-N_0^TN_0\right),\right.$$
$$\left.-(1-\alpha)\left(\varepsilon_7I-N_1^TN_1\right),-(1-\beta)\left(\varepsilon_8I-N_2^TN_2\right)\right\} \quad (14)$$

where,

$$\Psi_1=\dot{P}_1^T+\left(A^T+A_1^T\right)P_2+P_2^T\left(A+A_1\right)+W_3^TA_1+A_1^TW_3+S$$
$$+\varepsilon_1E^TE+\left(\varepsilon_2+\varepsilon_4\right)E_1^TE_1+\left(\varepsilon_5+\varepsilon_6+\varepsilon_7\frac{1}{1-\alpha}+\varepsilon_8\frac{1}{1-\beta}\right)E_c^TE_c$$
$$\Psi_2=P_1^T-P_2^T+\left(A^T+A_1^T\right)P_3+A_1^TW_4,$$
$$\Psi_3=-P_3-P_3^T+U+h_{\max}\bar{A}_{1,\max}^TR_3\bar{A}_{1,\max}$$
$$\Phi_1=\left[W_1^T+P_1^T \quad W_3^T+P_2^T\right],\qquad \Phi_2=\left[W_2^T \quad W_4^T+P_3^T\right]$$
$$R=R^T=\begin{bmatrix}R_1 & R_2\\ R_2^T & R_3\end{bmatrix}>0,$$
$$\tilde{C}^T\tilde{C}=C_0^TC_0+\frac{1}{1-\alpha}C_1^TC_1+\frac{1}{1-\beta}C_2^TC_2$$

Also suppose that $R_3(t)$ *satisfies the following matrix inequality,*

$$\dot{\bar{A}}_{1,\max}^TR_3\bar{A}_{1,\max}+\bar{A}_{1,\max}^TR_3\dot{\bar{A}}_{1,\max}+\bar{A}_{1,\max}^T\dot{R}_3\bar{A}_{1,\max}<0 \quad (15)$$

Under these conditions, the cost function (10) satisfies $J<0$ *for all nonzero* $w\in L_2^q[0,T]$, *and for any value of the positive delay* $g(t)$, $t\in[0,T]$.

Remark 1. Concerning a practical numerical procedure for finding the solutions to the above inequalities, the time derivatives of $P_1(t)$, $S(t)$, $U(t)$ and $R_3(t)$ can be approximated using a difference scheme (backward, forward or trapezoidal). Once the disctretization time-step is chosen, inequalities *(11)* and *(15)* reduce to a set of LMIs [7] to be solved at each iteration in time with terminal conditions $P_1(T)=P_{1T}$, $S(T)=S_T$, $U(T)=U_T$ and $R_3(T)=R_{3T}$. Here, the choice of S_T, U_T and R_{3T} is arbitrary provided that the conditions of the theorem are satisfied.

The following assumption [8] is needed to enable the application of Lyapunov's second method for stability of neutral systems in the infinite-horizon case (only uncertainties $F_g(t)$, $F(t)$, $F_1(t)$ and $F_c(t)$, and delays $g(t)$ and $h(t)$ are time-varying):

Assumption 1: The difference operator $D:\ C_n(-\infty,0]\times R\rightarrow R^n$, given by $D(x_t,t)=x(t)-\bar{G}(t)x(t-g(t))$, is delay-independently stable (i.e., the homogeneous difference equation $Dx_t=0$ is asymptotically stable irrespective of the delay g).

Theorem 2. (Bounded Real Lemma : infinite-horizon case) *Consider the time-invariant version of system (1)-(2) as defined above. For a given* $\gamma>0$, *suppose there exist* $n\times n$ *-matrices:* $P_1>0$, P_2, P_3, $S=S^T>0$, $U=U^T>0$, W_1, W_2, W_3, W_4 *and* $R_1=R_1^T$, R_2, $R_3=R_3^T>0$, *and positive scalars* ε_i; $i=1...8$, *which satisfy the following linear matrix inequality:*

$$\begin{bmatrix}\tilde{\Omega}_1 & \tilde{\Omega}_2\\ \tilde{\Omega}_2^T & \tilde{\Omega}_3\end{bmatrix}<0 \quad (16)$$

where, (16) has the same expression as (11) in Theorem 1, except that all entries are time-invariant.
Under these conditions, system (1) is globally uniformly asymptotically stable (g.u.a.s), and the cost function

$$J_0=\int_0^\infty\left[z^T(t)z(t)-\gamma^2w^T(t)w(t)\right]dt \quad (17)$$

satisfies $J_0<0$, *for all nonzero* $w\in L_2^q[0,\infty)$, *and for any value of the positive delay* g.

5. Numerical example.

Consider the uncertain time-delay system *(1)* with:

$$A=\begin{bmatrix}-2 & 0\\ 1 & -3\end{bmatrix},\ A_1=\begin{bmatrix}-1 & 0\\ -0.8 & -1\end{bmatrix},\ B_1=\begin{bmatrix}-0.5\\ 1\end{bmatrix},\ C_0=\begin{bmatrix}1 & 0\end{bmatrix}$$

$L=E=diag\left\{\sqrt{0.2},\sqrt{0.2}\right\}$ and $L_1=E_1=diag\left\{\sqrt{0.2},\sqrt{0.2}\right\}$ and assume that all the other matrices in *(1)* are zero. Matrix $\bar{A}_{1,\max}$ is chosen as $\bar{A}_{1,\max}=\begin{bmatrix}-2.25 & -1\\ -1 & -1.45\end{bmatrix}$.

The same example was used in [9] to compare the method developed there with previous results while using $h_{\max} = 0.5351$. The application of Theorem 6.1 of this paper with $\alpha = 0$ and $\beta = 0$ guarantees robust stability for $h_{\max} = 3.4232$, once again proving that the approach adopted here is much less conservative despite the presence of parametric uncertainties. In [10], for a rate of change $\alpha = 0.9$, robust stability is guaranteed for $h_{\max} = 0.3151$. Using Theorem 2 with $\alpha = 0.9$, robust stability is found to be guaranteed for $h_{\max} = 0.8265$. Finally, assuming the following value for $G = \begin{bmatrix} -0.8 & 0 \\ 0.8 & -0.1 \end{bmatrix}$ and $\alpha = 0$, the system is still stable for $h_{\max} = 0.2008$. For $h = 0.1$, a minimum value of $\gamma = 1.0098$ is achieved.

6. Conclusion

The problem formulation employed here is believed to be the most general considered so far with reference to the class of neutral systems. Such systems are highly relevant to applications in process industry and are likely to be employed there. As demonstrated, the design conditions derived in this paper are least restrictive as compared with previous works. A major innovation of the approach adopted here is the explicit dependence of the feedback law on the value of the time-varying time-delay as well as on its rate of change. Ongoing work is concerned with the construction of robust observers for neutral systems and the combined design of output-feedback.

Appendix

The following lemmas will prove helpful in the sequel:

Lemma 1: [9]. *Let the function* $v(t) = \int_{a(t)}^{b(t)} \int_{t-\theta}^{t} f(s) ds d\theta$. *Then* $v(t)$ *is a solution of the differential equation,*

$$\frac{dv(t)}{dt} = \big(b(t) - a(t)\big) f(t) - \big(1 - \dot{b}(t)\big) \int_{t-b(t)}^{t-a(t)} f(s) ds + \big(\dot{b}(t) - \dot{a}(t)\big) \int_{t-a(t)}^{t} f(s) ds.$$

Lemma 2: [11]. *Let* A, L, E *and* F *be real matrices (possibly time-varying) of appropriate dimensions, with* F *satisfying* $FF^T \le I$ *; then the following holds:*

1- For any scalar $\varepsilon > 0$,

$$P^T LFE + E^T F^T L^T P \le \varepsilon^{-1} P^T LL^T P + \varepsilon E^T E$$

2- For any matrix $P > 0$ *and any scalar* $\varepsilon > 0$ *such that* $\varepsilon I - EPE^T > 0$,

$$(A + LFE) P (A + LFE)^T \le APA^T + APE^T \left(\varepsilon I - EPE^T\right)^{-1} EPA^T + \varepsilon LL^T$$

3- For any matrix $P > 0$ *and any scalar* $\varepsilon > 0$ *such that* $P - \varepsilon LL^T > 0$,

$$(A + LFE)^T P^{-1} (A + LFE) \le A^T \left(P - \varepsilon LL^T\right)^{-1} A + \varepsilon^{-1} E^T E$$

Remark 2. Statement 3 in Lemma 2 as applied to $\bar{A}_1$ can be used to choose bound $\bar{A}_{1,\max}$ of *(9)*.

Lemma 3: [12]. *Assume that* $a(s) \in R^n$ *and* $b(s) \in R^m$ *are integrable over* $s \in \Omega$. *Then, for any positive definite matrix* $R \in R^{n\times m}$ *and any matrix* $M \in R^{m\times m}$, *the following holds:*

$$-2\int_{\Omega} b^T(s) a(s) ds \le \int_{\Omega} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} R & RM \\ M^T R & \Gamma \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds \tag{18}$$

where $\Gamma = \left(M^T R + I\right) R^{-1} (MR + I)$.

Proof of Theorem 1. Following [6], the unforced system ($u(t) = 0$) is first considered, and equation *(1)* is written in its equivalent descriptor form:

$$\dot{x}(t) = y(t), \quad y(t) = \bar{G}(t) y\big(t - g(t)\big) + \bar{A}(t) x(t) + \bar{A}_1(t) x\big(t - h(t)\big) + B_1(t) w(t) \tag{19}$$

Using $x\big(t - h(t)\big) = x(t) - \int_{t-h(t)}^{t} \dot{x}(s) ds$ (Liebniz-Newton) permits to re-write *(19)* yet in a more tractable form without introducing additional dynamics [13]:

$$\dot{x}(t) = y(t), \; 0 = -y(t) + \bar{G}(t) y\big(t - g(t)\big) + \left[\bar{A}(t) + \bar{A}_1(t)\right] x(t) - \bar{A}_1(t) \int_{t-h(t)}^{t} y(s) ds + B_1(t) w(t) \tag{20}$$

so that for $E = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$, the augmented system is:

$$E \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ \bar{A}(t) + \bar{A}_1(t) & -I \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{G}(t) \end{bmatrix} y\big(t - g(t)\big) - \begin{bmatrix} 0 \\ \bar{A}_1(t) \end{bmatrix} \int_{t-h(t)}^{t} y(s) ds + \begin{bmatrix} 0 \\ B_1(t) \end{bmatrix} w(t) \tag{21}$$

A time-varying generalization of the Lyapunov-Krasovskii functional, introduced in [6], will be used here:

$$V(t) = V_0(t) + V_1(t) + V_2(t) + V_3(t) \tag{22}$$

where,

$$V_0(t) = \left[x^T(t) \quad y^T(t)\right] EP(t) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = x^T(t) P_1(t) x(t) \tag{23}$$

$$V_1(t) = \int_{t-h(t)}^{t} x^T(\tau) S(t) x(\tau) d\tau \tag{24}$$

$$V_2(t)=\int_{t-g(t)}^{t} y^T(\tau)U(t)y(\tau)d\tau \qquad (25)$$

$$V_3(t)=\int_{-h_{\max}}^{0}\int_{t+\theta}^{t} y^T(s)\bar{A}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)y(s)dsd\theta \qquad (26)$$

with $P(t)=\begin{bmatrix} P_1(t) & 0 \\ P_2(t) & P_3(t)\end{bmatrix}$, $P_1(t)>0$, $U(t)=U^T(t)>0$,

$$S(t)=S^T(t)>0, \quad R_3(t)=R_3^T(t)>0 \qquad (27)$$

The performance index J of *(10)* can be written as,

$$J=x^T(T)P_{1T}x(T)+\int_0^T\left[z^T(t)z(t)-\gamma^2w^T(t)w(t)+\dot{V}(t)\right]dt$$
$$+V(0)-V_0(T)-V_1(T)-V_2(T)-V_3(T) \quad (28)$$

Since $V(0)=0$, $V_0(T)=x^T(T)P_{1T}x(T)$, and $V_i(T)\geq 0$; $i=1...3$, then $J<0$ is guaranteed if,

$$\bar{J}=\int_0^T\left[z^T(t)z(t)-\gamma^2w^T(t)w(t)+\dot{V}(t)\right]dt<0 \qquad (29)$$

Differentiating *(23)* and using *(21)*, yields,

$$\frac{dV_0(t)}{dt}=2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\begin{bmatrix}0 & I \\ \bar{A}(t)+\bar{A}_1(t) & -I\end{bmatrix}\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$$
$$+2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\begin{bmatrix}0\\ \bar{G}(t)\end{bmatrix}y(t-g(t))$$
$$+\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}\left\{2P^T(t)\begin{bmatrix}0\\B_1(t)\end{bmatrix}w(t)+\dot{P}^T(t)\begin{bmatrix}x(t)\\0\end{bmatrix}\right\}$$
$$-2\int_{t-h(t)}^{t}\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\begin{bmatrix}0\\ \bar{A}_1(t)\end{bmatrix}y(s)ds \qquad (30)$$

A bound for the last term of *(30)* will be derived as follows: Define

$$\eta(t)=-2\int_{t-h(t)}^{t}\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\begin{bmatrix}0\\ \bar{A}_1(t)\end{bmatrix}y(s)ds \qquad (31)$$

Using Lemma 3 with:

$$a(s)=\begin{bmatrix}0\\ \bar{A}_1(t)\end{bmatrix}y(s), b(s)=P(t)\begin{bmatrix}x(t)\\y(t)\end{bmatrix}, \Omega=\left[t-h(t),t\right]$$

gives:

$$\eta(t)\leq\int_{t-h(t)}^{t}\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\left(M^T(t)R(t)+I\right)$$
$$\times R^{-1}(t)\left(M(t)R(t)+I\right)P(t)\begin{bmatrix}x(t)\\y(t)\end{bmatrix}ds$$
$$+2\int_{t-h(t)}^{t}y^T(s)ds\begin{bmatrix}0 & \bar{A}_1^T(t)\end{bmatrix}R(t)M(t)P(t)\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$$
$$+\int_{t-h(t)}^{t}y^T(s)\begin{bmatrix}0 & \bar{A}_1^T(t)\end{bmatrix}R(t)\begin{bmatrix}0\\ \bar{A}_1(t)\end{bmatrix}y(s)ds$$
$$\leq h_{\max}\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\left(M^T(t)R(t)+I\right)$$
$$\times R^{-1}(t)\left(M(t)R(t)+I\right)P(t)\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$$
$$+2\left[x^T(t)-x^T(t-h(t))\right]\begin{bmatrix}0 & \bar{A}_1^T(t)\end{bmatrix}R(t)M(t)P(t)\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$$
$$+\int_{t-h_{\max}}^{t}y^T(s)\begin{bmatrix}0 & \bar{A}_1^T(t)\end{bmatrix}R(t)\begin{bmatrix}0\\ \bar{A}_1(t)\end{bmatrix}y(s)ds \qquad (32)$$

as $R(t)>0$. Differentiating $V_1(t)$ and $V_2(t)$ of *(24)* and *(25)*,

$$\frac{dV_1(t)}{dt}=x^T(t)S(t)x(t)$$
$$-\left(1-\dot{h}(t)\right)x^T(t-h(t))S(t)x(t-h(t))+\int_{t-h(t)}^{t}x^T(\tau)\dot{S}(t)x(\tau)d\tau$$

$$\frac{dV_2(t)}{dt}=y^T(t)U(t)y(t)$$
$$-\left(1-\dot{g}(t)\right)y^T(t-g(t))U(t)y(t-g(t))+\int_{t-g(t)}^{t}y^T(\tau)\dot{U}(t)y(\tau)d\tau$$

and using the assumptions $S(t)>0$, $\dot{S}(t)\leq 0$, $U(t)>0$ and $\dot{U}(t)\leq 0$, and the bounds *(8)*,

$$\frac{dV_1(t)}{dt}\leq x^T(t)S(t)x(t)$$
$$-(1-\alpha)x^T(t-h(t))S(t)x(t-h(t)) \quad (33)$$

$$\frac{dV_2(t)}{dt}\leq y^T(t)U(t)y(t)$$
$$-(1-\beta)y^T(t-g(t))U(t)y(t-g(t)) \quad (34)$$

Applying Lemma 1 to $V_3(t)$,

$$\frac{dV_3(t)}{dt}=h_{\max}y^T(t)\bar{A}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)y(t)$$
$$-\int_{t-h_{\max}}^{t}y^T(s)\bar{A}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)y(s)ds$$
$$+\int_{-h_{\max}}^{0}\int_{t+\theta}^{t}y^T(s)\frac{d}{dt}\left[\bar{A}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)\right]y(s)dsd\theta \; (35)$$

If,

$$\frac{d}{dt}\left[\bar{A}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)\right]=\dot{\bar{A}}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)$$
$$+\bar{A}_{1,\max}^T(t)R_3(t)\dot{\bar{A}}_{1,\max}(t)+\bar{A}_{1,\max}^T(t)\dot{R}_3(t)\bar{A}_{1,\max}(t)<0 \quad (36)$$

then,

$$\frac{dV_3(t)}{dt}<h_{\max}y^T(t)\bar{A}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)y(t)$$
$$-\int_{t-h_{\max}}^{t}y^T(s)\bar{A}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)y(s)ds \quad (37)$$

Notice that,

$$\int_0^T z^T(t)z(t)dt=$$
$$\int_0^T\left[\bar{C}_0(t)x(t)+D(t)w(t)\right]^T\left[\bar{C}_0(t)x(t)+D(t)w(t)\right]dt$$
$$+\int_{-h(t)}^{T-h(t)}\left[\bar{C}_1(t)x(t)\right]^T\left[\bar{C}_1(t)x(t)\right]\frac{1}{1-\dot{h}(t)}dt$$
$$+\int_{-g(t)}^{T-g(t)}\left[\bar{C}_2(t)x(t)\right]^T\left[\bar{C}_2(t)x(t)\right]\frac{1}{1-\dot{g}(t)}dt$$

$$
\begin{aligned}
&= \int_0^T \left[\bar{C}_0(t)x(t)+D(t)w(t)\right]^T \left[\bar{C}_0(t)x(t)+D(t)w(t)\right]dt \\
&\quad + \int_0^{T-h(t)} \left[\bar{C}_1(t)x(t)\right]^T \left[\bar{C}_1(t)x(t)\right]\frac{1}{1-\dot{h}(t)}dt \\
&\quad + \int_0^{T-g(t)} \left[\bar{C}_2(t)x(t)\right]^T \left[\bar{C}_2(t)x(t)\right]\frac{1}{1-\dot{g}(t)}dt \\
&\le \int_0^T \left[\bar{C}_0(t)x(t)+D(t)w(t)\right]^T \left[\bar{C}_0(t)x(t)+D(t)w(t)\right]dt \\
&\quad + \int_0^{T} \left[\bar{C}_1(t)x(t)\right]^T \left[\bar{C}_1(t)x(t)\right]\frac{1}{1-\alpha}dt \\
&\quad + \int_0^{T} \left[\bar{C}_2(t)x(t)\right]^T \left[\bar{C}_2(t)x(t)\right]\frac{1}{1-\beta}dt \qquad (38)
\end{aligned}
$$

Employing *(30)*, *(32)*, *(33)*, *(34)*, *(37)* and *(38)* in *(29)*, while noticing that, since $R_3(t)>0$ and from bound *(9)*,

$$
\begin{aligned}
&\int_{t-h_{\max}}^{t} y^T(s)\left[0 \quad \bar{A}_1^T(t)\right]R(t)\begin{bmatrix}0\\ \bar{A}_1(t)\end{bmatrix}y(s)\,ds \\
&\qquad - \int_{t-h_{\max}}^{t} y^T(s)\bar{A}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)y(s)\,ds \\
&= \int_{t-h_{\max}}^{t} y^T(s)\left[\bar{A}_1^T(t)R_3(t)\bar{A}_1(t)-\bar{A}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)\right] \\
&\qquad \times y(s)\,ds \le 0
\end{aligned}
$$

and finally applying Lemma 2 results in a bound for $\bar{J}$ which has the following form:

$\bar{J} \le \int_0^T \xi^T(t)\left[\Omega_1(t)-\Omega_2(t)\Omega_3^{-1}(t)\Omega_2^T(t)\right]\xi(t)\,dt$, where $\xi(t)$ is the aggregated state vector, and $\Omega_1(t)$, $\Omega_2(t)$ and $\Omega_3(t)$ are as in *(12)*, *(13)* and *(14)*, respectively.

Sufficient conditions for guaranteeing that $\bar{J}<0$ (and so $J<0$) can thus be stated as follows:

1.) $\Omega_1(t)-\Omega_2(t)\Omega_3^{-1}(t)\Omega_2^T(t)<0$.

2.) $\Omega_3(t)<0$ (assumptions of Lemma 2).

3.) $\dot{\bar{A}}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)+\bar{A}_{1,\max}^T(t)R_3(t)\dot{\bar{A}}_{1,\max}(t)$
$+\bar{A}_{1,\max}^T(t)\dot{R}_3(t)\bar{A}_{1,\max}(t)<0$ (used for obtaining *(37)*).

By the Schur complements lemma, conditions 1.) and 2.) are equivalent to assumption *(11)* of the Theorem, and condition 3.) is assumption *(15)* of the Theorem. QED

Proof of Theorem2. From *(19)* and the fact that x and w are square integrable on $[0,\infty)$, it follows that $D(y_t)\in L_2^n[0,\infty)$. Under Assumption 1, the latter implies that $y_t \in L_2^n[0,\infty)$. By the application of the Schur complements lemma to *(16)*, it is easy to show that $\dot{V}<0$, which along with Assumption 1 guarantees the global uniform asymptotic stability of system *(1)*, and also implies that $z\in L_2^p[0,\infty)$ and that J_0 is well defined. Close inspection of *(15)* further implies that this condition is redundant. The proof of $J_0<0$ then follows exactly like that of Theorem 1. QED

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