Delay-dependent robust stabilization of uncertain neutral systems with saturating actuators

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Abstract

The robust stabilization problem for neutral systems with saturating actuators is addressed here. The systems considered are continuous-time with parametric uncertainties entering all the matrices in the system representation. A saturating control law is designed and a region is specified in which the stability of the closed-loop system is ensured. The Lyapunov-Krasovskii functional is employed. The design is dependent on the time-delay and its rate of change. The controller is constructed in terms of the solution to a set of matrix inequalities by employing a descriptor model transformation of the system.

1. Introduction

In recent years, much work has been devoted to the analysis and synthesis of controllers for state-delayed systems with or without parametric uncertainties; see [1] for earlier work in this area. This interest is strongly motivated by the fact that delays and uncertainties are the two most important causes of instability. Furthermore, both delays and uncertainties occur frequently in the chemical and process industries, which provides another reason for the study of new and less conservative stability conditions.

The control design problem for the general case of neutral (descriptor) systems , in which the time-delays can appear both in the state and its derivative, has so far obtained relatively little attention [2]. Unlike simple retarded systems, neutral systems are particularly sensitive to delays and can be easily destabilized [3,4].

To the best knowledge of the authors, stabilization of neutral systems with actuator saturation has only been studied in [5]. However, the controller designed in [5] is delay-independent (which is relatively conservative), the system representation includes no uncertainties, and the delays are considered known and time-invariant.

In this context, the contribution of this paper is as follows:

- A delay-dependent robust stabilization problem is addressed here for neutral systems in full generality, with actuators constraints and norm-bounded parametric uncertainties entering all the matrices in the system representation.
- In attempt to make the approach least conservative, a descriptor model transformation and a Lyapunov-Krasovskii functional are employed here, as in [6].
- The least conservative representation of actuator saturation, as shown in [7], is used here.
- The value of the time-delay as well as its rate of change are taken into account in the design method presented which further reduces the conservativeness of the approach.

2. Adopted notation

The following notation is adopted. For any matrix A, the expressions A^T and $diag\{A\}$ denote the transpose of A and the diagonal of A, respectively. R^n is the n dimensional Euclidean space with vector norm $\|\,.\,\|$, $R^{n\times m}$ is the set of all $n\times m$ real matrices, and for any matrix $P\in R^{n\times n}$, the inequality $P>0$, signifies that P is symmetric and positive definite. The standard notation of $L_2^q[a,b]$ is adopted for the space of all functions $f:R\to R^q$ which are Lebesque integrable in the square over the interval $[a,b]$, with the standard norm $\|\,.\,\|_{L_2}$. The symbol $X_n[a,b]$ is used to denote the space of all continuous functions $x:[a,b]\to R^n$, with the standard supremum norm $\|\,.\,\|_\infty$. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximal and minimal eigenvalue of matrix P, respectively. The symbol $C_{d,n}=C([-d,0],R^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-d,0]$ into R^n with the topology of uniform convergence; $\|\phi\|_c=\sup_{-d\le t\le 0}\|\phi(t)\|$ is the norm of a function $\phi\in C_{d,n}$. The set $C_{d,n}^w$ is defined by $C_{d,n}^w=\{\phi\in C_{d,n};\ \|\phi\|_c<w\}$, where w is a positive real

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number. For any vector $v \in R^m$, $sat(v_{(i)}) = sign(v_{(i)}) \min(u_{0(i)}, |v(i)|)$, $u_{0(i)} > 0$, $i = 1,...,m$. For any $t \geq t_0$, x_t denotes the restriction of x to the interval $[t-d, t]$ translated to $[-d, 0]$, that is, $x_t(\psi) = x(t+\psi)$, for all $\psi \in [-d, 0]$.

3. Problem statement

Consider the following system:

$$\dot{x}(t) - \bar{G}\dot{x}(t-g(t)) = \bar{A}x(t) + \bar{A}_1 x(t-h(t)) + \bar{B}u(t) \qquad (1)$$

where the delays h and g are assumed to be some unknown functions of time and are bounded as:

$$0 \leq h(t) \leq h_{\max} \text{ and } 0 \leq g(t) < \infty \quad \text{for all } t \geq 0 \qquad (2)$$

and for some given positive constant $h_{\max}$, $x(t) \in R^n$ is the system state vector and $u(t) \in R^m$ is the control input.
The initial conditions for system *(1)* are given as:

$$x(t_0+\psi) = \phi(\psi),\ \forall \psi \in [-d_{\max}, 0],\ (t_0, \phi) \in R^+ \times C^w_{d_{\max},n}$$

with $d_{\max} = \max(h(t), g(t)),\ \forall t \geq 0$

The uncertain matrices $\bar{G}$, $\bar{A}$, $\bar{A}_1$ and $\bar{B}$ are given by:

$$\bar{G} = G + \Delta G,\ \bar{A} = A + \Delta A,\ \bar{A}_1 = A_1 + \Delta A_1,\ \bar{B} = B + \Delta B \quad (3)$$

where the matrices G, A, A_1 and B are real, and are assumed to be known exactly. The matrices ΔG, ΔA, ΔA_1 and ΔB are real-valued, represent the norm-bounded parameter uncertainties, and are assumed to be of the following form:

$$\Delta G = L_g F_g E_g,\quad \begin{bmatrix} \Delta A \\ \Delta B \end{bmatrix} = LF \begin{bmatrix} E \\ E_b \end{bmatrix},\quad \Delta A_1 = L_1 F_1 E_1 \qquad (4)$$

where $F_g \in R^{i_g \times j_g}$, $F \in R^{i \times j}$ and $F_1 \in R^{i_1 \times j_1}$ are real, uncertain, possibly time-varying matrices with Lebesgue measurable entries which, additionally, meet the following requirements: $F_g F_g^T \leq I$, $FF^T \leq I$ and $F_1 F_1^T \leq I$. The matrices L_g, L, L_1, E_g, E, E_1 and E_b are known and real, and characterize the way in which the uncertain parameters of F_g, F and F_1 enter the nominal matrices G, A, A_1 and B.

Additionally, delays h and g are assumed to be continuously differentiable, with their respective rates of change bounded as follows:

$$0 \leq \dot{h}(t) \leq \beta_h < 1,\quad 0 \leq \dot{g}(t) \leq \beta_g < 1 \quad \text{for } t \geq 0 \qquad (5)$$

where β_h and β_g are given positive constants.

Also, $\bar{A}_1$ is bounded as follows:

$$\bar{A}_1^T \bar{A}_1 \leq \bar{A}_{1,\max}^T \bar{A}_{1,\max} \qquad (6)$$

where $\bar{A}_{1,\max}$ is constant and known.

The following is assumed:

Assumption 1: $(A + A_1, B)$ is stabilizable.

Assumption 2: The input vector is subject to amplitude constraints, i.e. $u \in U_0 \subset R^m$, with

$$U_0 = \left\{ u \in R^m\ ;\ -u_{0(i)} \leq u_{(i)} \leq u_{0(i)},\ u_{0(i)} > 0,\ i = 1...m \right\} \quad (7)$$

The following additional assumption [9] is needed to enable the application of Lyapunov's second method for the stability of neutral systems:

Assumption 3: The difference operator $D: \mathrm{X}_n(-\infty, 0] \times R \to R^n$, given by $D(x_t, t) = x(t) - \bar{G}(t)x(t-g(t))$, is delay-independently stable (i.e., the homogeneous difference equation $Dx_t = 0$ is asymptotically stable irrespective of the delay g).

The robust stabilization problem with saturating actuators:
Find matrix $K \in R^{m \times n}$ and a set of initial conditions $S_0 \in R^n$ such that the asymptotic stability of the closed-loop system :

$$\dot{x}(t) - \bar{G}\dot{x}(t-g(t)) = \bar{A}x(t) + \bar{A}_1 x(t-h(t)) + \bar{B}sat(Kx(t)) \qquad (8)$$

is ensured, that is, $\forall \phi(\psi) \in S_0$, $\forall \psi \in [-d_{\max}, 0]$, system *(8)* is locally asymptotically stable.

Remark 1. Since no assumption is made about the stability of the open-loop system, the problem at hand is that of local stability. The global asymptotic stability of system *(8)* can be addressed only if the open-loop system is assumed stable.

4. Main results

The following theorem delivers the main result of this paper.

Theorem 1. *Consider the system (8). Suppose that there exist* $n \times n$*-matrices:* $P_1 > 0$, P_2, P_3, $S = S^T > 0$, $U = U^T > 0$, W_1, W_2, W_3, W_4, *and* $R_1 = R_1^T$, R_2, $R_3 = R_3^T > 0$, *a matrix* $K \in R^{m \times n}$, *a vector* $\alpha_l \in R^m$, *a positive scalar* γ *and positive scalars* ε_i; $i = 1...4$, *which satisfy the following matrix inequalities:*

$$\begin{bmatrix} \Omega_{1j} & \Omega_2 \\ \Omega_2^T & \Omega_3 \end{bmatrix} < 0, \qquad \forall j = 1,...,2^m \qquad (9)$$

$$\begin{bmatrix} P_1 & \alpha_{l(i)} K_{(i)}^T \\ \alpha_{l(i)} K_{(i)} & \gamma u_{0(i)}^2 \end{bmatrix} \geq 0, \quad \forall i = 1,...,m \qquad (10)$$

$$0<\alpha_{l(i)}\leq 1, \qquad \forall i=1,\dots,m \tag{11}$$

with,

$$\Omega_{1j}=\begin{bmatrix} \Psi_{1j} & \Psi_{2j} & h_{\max}\Phi_1 & -W_3^TA_1-\varepsilon_4E_1^TE_1 & P_2^TG \\ \Psi_{2j}^T & \Psi_3 & h_{\max}\Phi_2 & -W_4^TA_1 & P_3^TG \\ h_{\max}\Phi_1^T & h_{\max}\Phi_2^T & -h_{\max}R & 0 & 0 \\ -A_1^TW_3-\varepsilon_4E_1^TE_1 & -A_1^TW_4 & 0 & -(1-\beta_h)S+\varepsilon_4E_1^TE_1 & 0 \\ G^TP_2 & G^TP_3 & 0 & 0 & -(1-\beta_g)U+\varepsilon_3E_g^TE_g \end{bmatrix} \tag{12}$$

$$\Omega_2=\begin{bmatrix} P_2^TL & P_2^TL_1 & P_2^TL_g & W_3^TL_1 \\ P_3^TL & P_3^TL_1 & P_3^TL_g & W_4^TL_1 \\ & 0_{3\times 4} & & \end{bmatrix} \tag{13}$$

$$\Omega_3=diag\left\{-\varepsilon_1 I,-\varepsilon_2 I,-\varepsilon_3 I,-\varepsilon_4 I\right\} \tag{14}$$

where,

$$\begin{aligned}\Psi_{1j}=&\left(A+A_1+B\Gamma_j(\alpha_l)K\right)^TP_2+P_2^T\left(A+A_1+B\Gamma_j(\alpha_l)K\right)\\ &+W_3^TA_1+A_1^TW_3+S+(\varepsilon_2+\varepsilon_4)E_1^TE_1\\ &+\varepsilon_1\left(E+E_b\Gamma_j(\alpha_l)K\right)^T\left(E+E_b\Gamma_j(\alpha_l)K\right)\end{aligned}$$

$$\Psi_{2j}=P_1^T-P_2^T+\left(A+A_1+B\Gamma_j(\alpha_l)K\right)^TP_3+A_1^TW_4,$$

$$\Psi_3=-P_3-P_3^T+U+h_{\max}\bar{A}_{1,\max}^TR_3\bar{A}_{1,\max}$$

$$\Phi_1=\left[W_1^T+P_1^T \quad W_3^T+P_2^T\right], \qquad \Phi_2=\left[W_2^T \quad W_4^T+P_3^T\right]$$

$$R=R^T=\begin{bmatrix}R_1 & R_2\\ R_2^T & R_3\end{bmatrix}>0$$

Under these conditions, system (8) is asymptotically stable for any initial condition ϕ belonging to the ball $\Phi(\sigma)$, where:

$$\Phi(\sigma)=\left\{\phi\in C^w_{d_{\max},n};\ \|\phi\|_c^2\leq\sigma\right\} \tag{15}$$

with $\sigma=\dfrac{1}{\gamma\pi_2}$

where,

$$\begin{aligned}\pi_2=\max\Bigg\{&\lambda_{\max}(P_1)+2\frac{h_{\max}}{(1-\beta_h)}\lambda_{\max}(S),\\ &2h_{\max}^2\lambda_{\max}(S)+\frac{1}{(1-\beta_g)}\lambda_{\max}(U)+h_{\max}\lambda_{\max}\left(\bar{A}_{1,\max}^TR_3\bar{A}_{1,\max}\right)\Bigg\}\end{aligned} \tag{16}$$

Remark 2. In [7], a comparison between the different representations of saturations used in the literature was made , concluding that the difference inclusions method used in this paper (see proof of Theorem 1 in Appendix for the this form of representation) leads to the least conservative design.

Remark 3. In [5], the proposed design is delay-independent in both the neutral and the retarded delays, while in Theorem1 of this paper the presented controller is delay-independent in the neutral delay and delay-dependent in the retarded delay. This gives a less conservative design than in {5]. The reason for choosing a delay-independent design for the neutral delay is that unlike simple retarded systems, neutral systems are particularly sensitive to delays and can be easily destabilized [3,4]. Also in [5], only the uncertainty-free system is treated and the delays are assumed known and time-invariant, which makes the treatment in the present paper more general.

Remark 4. In [17], a delay-dependent design for retarded systems was presented. However, the descriptor system transformation used there introduces additional dynamics to the original system [14], and thus makes the design more conservative than the one proposed in the present paper. Also, the bounding methods used here (see proof of Theorem 1 in Appendix), reduce further the conservativeness as compared to [17].

Remark 5. The main difficulty in the application of Theorem 1 is the presence of some nonlinearities in the variables P_2, P_3, α_l and K. This can be overcome by using some relaxation techniques, [5,16], i.e. by iterating with respect to α_l and K. Inequalities *(9)* and *(10)* then reduce to linear matrix inequalities (LMIs) in all the remaining variables. The latter are easily solved using the techniques of [8].

5. Conclusion

The problem formulation employed here is believed to be the first and the most general considered so far with reference to the delay-dependent robust stabilization of neutral systems with saturating actuators. Such systems are highly relevant to applications in process industry and are likely to be employed there. A major innovation of the approach adopted here is that the stabilizing control design is made dependent on both the value of the time-delay as well as on its rate of change. Ongoing work is concerned with the extension of the present work to the full robust H_∞ control problem.

Appendix

The following lemmas will prove helpful in the sequel:

Lemma 1: [10]. *Let the function* $z(t)=\int_{a(t)}^{b(t)}\int_{t-\theta}^{t}f(s)dsd\theta$. *Then* $z(t)$ *is a solution of the differential equation,*

$$\frac{dz(t)}{dt}=\left(b(t)-a(t)\right)f(t)-\left(1-\dot{b}(t)\right)\int_{t-b(t)}^{t-a(t)}f(s)ds$$

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$$+\left(\dot{b}(t)-\dot{a}(t)\right)\int_{t-a(t)}^{t} f(s)\,ds .$$

Lemma 2: [11]. *Let A, L, E and F be real matrices (possibly time-varying) of appropriate dimensions, with F satisfying $FF^T \le I$; then the following holds:*

1- For any scalar $\varepsilon > 0$,

$$P^T LFE + E^T F^T L^T P \le \varepsilon^{-1} P^T LL^T P + \varepsilon E^T E$$

2- For any matrix $P > 0$ and any scalar $\varepsilon > 0$ such that $\varepsilon I - EPE^T > 0$,

$$(A+LFE)P(A+LFE)^T \le APA^T + APE^T\left(\varepsilon I - EPE^T\right)^{-1} EPA^T + \varepsilon LL^T$$

3- For any matrix $P > 0$ and any scalar $\varepsilon > 0$ such that $P - \varepsilon LL^T > 0$,

$$(A+LFE)^T P^{-1}(A+LFE) \le A^T\left(P-\varepsilon LL^T\right)^{-1} A + \varepsilon^{-1} E^T E$$

Remark 6. Statement 3 in Lemma 2 as applied to $\bar{A}_1$ can be used to choose bound $\bar{A}_{1,\max}$ of *(6)*.

Lemma 3: [12]. *Assume that $a(s)\in R^n$ and $b(s)\in R^m$ are integrable over $s\in\Omega$. Then, for any positive definite matrix $R\in R^{n\times m}$ and any matrix $M\in R^{m\times m}$, the following holds:*

$$-2\int_\Omega b^T(s)a(s)\,ds \le \int_\Omega \begin{bmatrix} a(s) \\ b(s)\end{bmatrix}^T \begin{bmatrix} R & RM \\ M^T R & \Gamma \end{bmatrix}\begin{bmatrix} a(s) \\ b(s)\end{bmatrix} ds \qquad (17)$$

where $\Gamma = \left(M^T R + I\right)R^{-1}(MR+I)$.

Proof of Theorem 1. A locally equivalent polytopic representation for the nonlinear system *(8)* based on difference inclusions [13] is used, leading to,

$$\dot{x}(t)-\bar{G}\dot{x}(t-g(t)) = \left(\bar{A}+\bar{B}\Gamma(\alpha(x))K\right)x(t)+\bar{A}_1 x(t-h(t)) \qquad (18)$$

where $\Gamma(\alpha(x))$ is a diagonal matrix whose diagonal elements are defined for $i=1,\dots,m$ as:

$$\alpha_{(i)}(x) = \begin{cases} \dfrac{u_{0(i)}}{v_{(i)}} & \text{if} \quad v_{(i)} > u_{0(i)} \\ 1 & \text{if} \quad -u_{0(i)} \le v_{(i)} \le u_{0(i)} \\ -\dfrac{u_{0(i)}}{v_{(i)}} & \text{if} \quad v_{(i)} < -u_{0(i)} \end{cases} \qquad (19)$$

with $v_{(i)} = K_{(i)}x(t)$. By definition, one gets $0 < \alpha_{(i)}(x) \le 1,\ i=1,\dots,m,\ \forall x\in R^n$. Note that the scalar $\alpha_{(i)}(x)$ can be considered as an indicator of the saturation degree of the i^{th} entry of $u_{(i)}$. Hence, the smaller is $\alpha_{(i)}(x)$, the farther is x from the region of linearity $S(u_0, 1_m)$ defined by,

$$S(u_0,1_m) = \left\{x\in R^n;\ \left|v_{(i)}\right| \le u_{0(i)},\ \ i=1,\dots,m\right\} \qquad (20)$$

Since this paper addresses the problem of local stabilization, we have to limit x and therefore to consider a lower bound for $\alpha_{(i)}(x)$. Therefore, if we consider any compact set $S_c \subset R^n$ then for $x\in S_c$ the components of vector $\alpha(x)$ admit a lower bound denoted:

$$\alpha_{l(i)} = \min\left\{\alpha_{(i)}(x);\ x\in S_c\right\},\ \ i=1,\dots,m \qquad (21)$$

That means that for any $x\in S_c$, one gets,

$$0 < \alpha_{l(i)} \le \alpha_{(i)}(x) \le 1,\ \ \forall i=1,\dots,m \qquad (22)$$

which allow to define the vectors α_l . From this vector, we can define the following vertex matrices:

$$\bar{A}_j = \bar{A} + \bar{B}\Gamma_j(\alpha_l)K \qquad (23)$$

where $\Gamma_j(\alpha_l)$ is a diagonal matrix whose diagonal elements arbitrarly take the values 1 or $\alpha_{l(i)}$, $i=1,\dots,m$. Hence, if $x\in S_c$ then $\dot{x}$ can be determined from the following polytopic model:

$$\dot{x}(t)-\bar{G}\dot{x}(t-g(t)) = \sum_{j=1}^{2^m}\lambda_j\bar{A}_j x(t) + \bar{A}_1 x(t-h(t)) \qquad (24)$$

with $\sum_{j=1}^{2^m}\lambda_j = 1,\ \ \lambda_j \ge 0$.

Furthermore, it is important to note that the vector α_l allows to define the polyhedral set as:

$$S(u_0,\alpha_l) = \left\{x\in R^n;\ -\left|v_{(i)}\right| \le \frac{u_{0(i)}}{\alpha_{l(i)}},\ i=1,\dots,m\right\} \qquad (25)$$

Actually, the set $S(u_0,\alpha_l)$ contains S_c and corresponds to the maximal set in which model *(24)* represents system *(8)*. We consider as a convenient choice for the set S_c ellipsoids generically defined from a symmetric positive definite matrix P as,

$$\Psi\left(P_1,\gamma^{-1}\right) = \left\{x\in R^n;\ x(t)^T P_1 x(t) \le \gamma^{-1}\right\} \qquad (26)$$

The satisfaction of assumption *(10)* of the Theorem means that the ellipsoid *(26)* is included in the set $S(u_0,\alpha_l)$ defined in *(25)* where the vector α_l verifies *(11)*. Then, $\dot{x}(t)$ can be computed from the polytopic system *(24)*.

Equation *(24)* is written in its equivalent descriptor form:

$$\dot{x}(t) = y(t),\quad y(t) = \bar{G}y(t-g(t)) + \sum_{j=1}^{2^m}\lambda_j\bar{A}_j x(t) + \bar{A}_1 x(t-h(t)) \qquad (27)$$

Using $x(t-h(t)) = x(t) - \int_{t-h(t)}^{t} \dot{x}(s)ds$ (Liebniz-Newton) permits to re-write *(27)* yet in a more tractable form without introducing additional dynamics [14]:

$$\dot{x}(t) = y(t),\ 0 = -y(t) + \bar{G}y(t-g(t))$$

$$+\left[\sum_{j=1}^{2^m} \lambda_j \bar{A}_j + \bar{A}_1\right] x(t) - \bar{A}_1 \int_{t-h(t)}^{t} y(s)\,ds \quad (28)$$

so that for $E = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$, the augmented system is:

$$E\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ \sum_{j=1}^{2^m} \lambda_j \bar{A}_j + \bar{A}_1 & -I \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{G} \end{bmatrix} y(t-g(t))$$

$$-\begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix} \int_{t-h(t)}^{t} y(s)\,ds \quad (29)$$

The following Lyapunov-Krasovskii functional will be used here:

$$V = V_0 + V_1 + V_2 + V_3 \quad (30)$$

where,

$$V_0 = \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = x^T(t) P_1 x(t) \quad (31)$$

$$V_1 = \int_{t-h(t)}^{t} x^T(\tau) S x(\tau)\, d\tau \quad (32)$$

$$V_2 = \int_{t-g(t)}^{t} y^T(\tau) U y(\tau)\, d\tau \quad (33)$$

$$V_3 = \int_{-h_{\max}}^{0} \int_{t+\theta}^{t} y^T(s) \bar{A}_{1,\max}^T R_3 \bar{A}_{1,\max} y(s)\, ds\, d\theta \quad (34)$$

with, $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$, $P_1 > 0$, $U = U^T > 0$, $S = S^T > 0$,

$$R_3 = R_3^T > 0 \quad (35)$$

Differentiating *(31)* and using *(29)*, yields,

$$\frac{dV_0}{dt} = 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 & I \\ \sum_{j=1}^{2^m} \lambda_j \bar{A}_j + \bar{A}_1 & -I \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 \\ \bar{G} \end{bmatrix} y(t-g(t))$$

$$-2\int_{t-h(t)}^{t} \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix} y(s)\,ds \quad (36)$$

A bound for the last term of *(36)* will be derived as follows: Define

$$\eta(t) = -2\int_{t-h(t)}^{t} \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix} y(s)\,ds \quad (37)$$

Using Lemma 3 with $a(s) = \begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix} y(s)$, $b(s) = P\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and with $\Omega = [t-h(t), t]$, gives:

$$\eta(t) \le$$

$$\int_{t-h(t)}^{t} \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T (M^T R + I) R^{-1} (MR + I) P \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} ds$$

$$+2\int_{t-h(t)}^{t} y^T(s)\,ds \begin{bmatrix} 0 & \bar{A}_1^T \end{bmatrix} RMP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+\int_{t-h(t)}^{t} y^T(s) \begin{bmatrix} 0 & \bar{A}_1^T \end{bmatrix} R \begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix} y(s)\,ds$$

$$\le h_{\max} \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T (M^T R + I) R^{-1} (MR + I) P \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+2\begin{bmatrix} x^T(t) - x^T(t-h(t)) \end{bmatrix} \begin{bmatrix} 0 & \bar{A}_1^T \end{bmatrix} RMP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+\int_{t-h_{\max}}^{t} y^T(s) \begin{bmatrix} 0 & \bar{A}_1^T \end{bmatrix} R \begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix} y(s)\,ds \quad (38)$$

as $R > 0$. Differentiating V_1 and V_2 of *(32)* and *(33)*,

$$\frac{dV_1}{dt} = x^T(t) S x(t) - (1 - \dot{h}(t)) x^T(t-h(t)) S x(t-h(t))$$

$$\frac{dV_2}{dt} = y^T(t) U y(t) - (1 - \dot{g}(t)) y^T(t-g(t)) U y(t-g(t))$$

and using the assumptions $S > 0$, $U > 0$, and the bounds *(5)*,

$$\frac{dV_1}{dt} \le x^T(t) S x(t) - (1 - \beta_h) x^T(t-h(t)) S x(t-h(t)) \quad (39)$$

$$\frac{dV_2}{dt} \le y^T(t) U y(t) - (1 - \beta_g) y^T(t-g(t)) U y(t-g(t)) \quad (40)$$

Applying Lemma 1 to V_3,

$$\frac{dV_3}{dt} = h_{\max} y^T(t) \bar{A}_{1,\max}^T R_3 \bar{A}_{1,\max} y(t)$$

$$-\int_{t-h_{\max}}^{t} y^T(s) \bar{A}_{1,\max}^T R_3 \bar{A}_{1,\max} y(s)\,ds \quad (41)$$

From *(27)* and the fact that x is square integrable on $[0,\infty)$, it follows that $D(y_t) \in L_2^n[0,\infty)$. Under Assumption 3, the latter implies that $y_t \in L_2^n[0,\infty)$.

Employing *(30)*, *(36)*, *(38)*, *(39)*, *(40)* and *(41)*, while noticing that, since $R_3 > 0$ and from bound *(6)*,

$$\int_{t-h_{\max}}^{t} y^T(s) \begin{bmatrix} 0 & \bar{A}_1^T \end{bmatrix} R \begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix} y(s)\,ds$$

$$-\int_{t-h_{\max}}^{t} y^T(s) \bar{A}_{1,\max}^T R_3 \bar{A}_{1,\max} y(s)\,ds$$

$$= \int_{t-h_{\max}}^{t} y^T(s) \begin{bmatrix} \bar{A}_1^T R_3 \bar{A}_1 - \bar{A}_{1,\max}^T R_3 \bar{A}_{1,\max} \end{bmatrix} y(s)\,ds \le 0$$

and finally applying Lemma 2 to terms in $\bar{G}$, $\bar{A}$, $\bar{A}_1$ and $\bar{B}$ results in a bound for $\dot{V}$ which has the following form: $\dot{V} \le \xi^T(t) \left[\Omega_1 - \Omega_2 \Omega_3^{-1} \Omega_2^T\right] \xi(t)$, where $\xi(t)$ is the aggregated state vector, and Ω_2 and Ω_3 are as in *(13)* and *(14)*, respectively. Concerning Ω_1, it has terms involving

$\sum_{j=1}^{2^m} \lambda_j(...)$, and so to guarantee $\dot{V} < 0$, we need to have $\Omega_1 - \Omega_2\Omega_3^{-1}\Omega_2^T < 0$. By convexity, the latter is guaranteed by having $\Omega_{1j} - \Omega_2\Omega_3^{-1}\Omega_2^T < 0$, for $j = 1,...,2^m$, where Ω_{1j} is as in *(12)*.

Sufficient conditions for guaranteeing that $\dot{V} < 0$ can thus be stated as follows:

1.) $\Omega_{1j} - \Omega_2\Omega_3^{-1}\Omega_2^T < 0$, $\qquad j = 1,...,2^m$

2.) $\Omega_3 < 0$ (assumptions of Lemma 2).

By the Schur complements lemma, conditions 1.) and 2.) are equivalent to assumption *(9)* of the Theorem.

Thus, there exists π_3 such that $\dot{V}(x_t) \le -\pi_3 \|x(t)\|^2$ and therefore one gets $V(x_t) \le V(x_{t_0})$ provided that the model *(24)* is valid, that is, provided that $x(t) \in S(u_0, \alpha_l)$.

Furthermore, following [18], the Lyapunov functional defined in *(30)* can be shown to satisfiy,

$$\pi_1 \|D\phi\|^2 \le V(\phi) \le \pi_2 \|\phi\|_c^2 \qquad (42)$$

with $\pi_1 = \lambda_{\min}(P_1)$ and

$$\pi_2 = \max\left\{\lambda_{\max}(P_1) + 2\frac{h_{\max}}{(1-\beta_h)}\lambda_{\max}(S),\right.$$

$$\left. 2h_{\max}^2\lambda_{\max}(S) + \frac{1}{(1-\beta_g)}\lambda_{\max}(U) + h_{\max}\lambda_{\max}\left(\bar{A}_{1,\max}^T R_3 \bar{A}_{1,\max}\right)\right\} \qquad (43)$$

Hence, for $\phi(\psi) \in \Phi,\ \psi \in [-2d_{\max}, 0]$, one gets,

$$x^T(t)P_1x(t) \le V(x_t) \le V(x_{t_0}) \le \gamma^{-1},\ \forall t \ge t_0 \qquad (44)$$

Therefore, for any initial condition ϕ in the ball $\Phi(\sigma)$ defined in *(15)*, the system *(8)* verifies the conditions of the Krasovskii Theorem [15] and $V(x_t)$ is a local strictly decreasing Lyapunov function. Thus the asymptotic stability of system *(8)* is ensured.

QED

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