Proceedings of the American Control Conference
Chicago, Illinois • June 2000

A Necessary Condition for Consistency of Noisy Experimental Closed-Loop Frequency-Response Data with SISO Coprime Factor Models

Benoit Boulet
Centre for Intelligent Machines, McGill University,
3480 University Street, Montréal, Québec, Canada H3A 2A7.

Abstract

This note introduces a necessary condition for the model/data consistency problem with perturbed coprime factorizations and closed-loop noisy frequency-response measurements. The necessary condition involves the computation of structured singular values of complex linear fractional transformations.

1 Introduction

Assume that a stabilizing controller providing sufficient damping was implemented on a lightly-damped or unstable SISO plant, allowing measurement of the closed-loop frequency response at N distinct frequencies. We want to find out whether a given nominal coprime-factor plant model combined with a factor perturbation from an uncertainty set could have produced the noisy closed-loop data. Solutions to the corresponding open-loop problem were given in [2] for coprime factor models, and [4] and [5] for more general linear fractional models. A necessary condition for the SISO closed-loop noisy coprime-factor model/data consistency problem is given. It consists of a test on N structured singular values of complex linear fractional transformations (LFTs). The upper and lower LFT of P by K are denoted $\mathcal{F}_L(P,K)$ and $\mathcal{F}_U(P,K)$. For a normed space $\mathcal{X}$, $\mathcal{B}\mathcal{X}$ denotes its open unit ball.

2 Necessary Condition for Consistency

Suppose we are given N nonzero noisy scalar frequency-response measurements $\{\phi_i\}_{i=1}^N$ in $\mathbb{C}$ corresponding to the distinct frequencies $\omega_1,\ldots,\omega_N$. For more generality, we consider a nominal left-coprime factorization of the plant where the factor $\tilde{\boldsymbol{M}}$ has dimensions $n\times n$ and the factor $\tilde{\boldsymbol{N}}$ has dimensions $n\times 1$, together with an output transfer matrix $\boldsymbol{C}$ of dimension $1\times n$. Thus the plant transfer function $\boldsymbol{g}$ is written as $\boldsymbol{g}=\boldsymbol{C}\tilde{\boldsymbol{M}}^{-1}\tilde{\boldsymbol{N}}$. This model allows us to treat scalar factorizations with $n=1$ and $\boldsymbol{C}=1$, but also our special LCF of

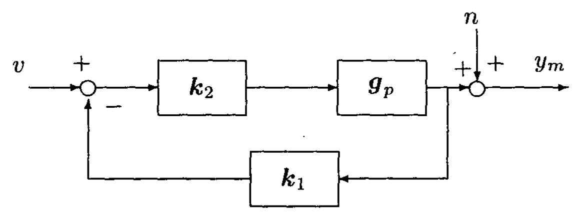

Figure 1: Noisy SISO feedback system.

LFSS [3]. Let the perturbed open-loop plant model $\boldsymbol{g}_p$ be expressed as a perturbed factorization with $\tilde{\boldsymbol{M}}_p$, $\tilde{\boldsymbol{N}}_p\in\mathcal{RH}_\infty$, that is, $\boldsymbol{g}_p=\boldsymbol{C}\tilde{\boldsymbol{M}}_p^{-1}\tilde{\boldsymbol{N}}_p$, where $\tilde{\boldsymbol{M}}_p=\tilde{\boldsymbol{M}}+\boldsymbol{\Delta}_M$, $\tilde{\boldsymbol{N}}_p=\tilde{\boldsymbol{N}}+\boldsymbol{\Delta}_N$, $\boldsymbol{\Delta}_M,\boldsymbol{\Delta}_N\in\mathcal{RH}_\infty$. Define the uncertainty matrix $\boldsymbol{\Delta}:=[\boldsymbol{\Delta}_N\ -\boldsymbol{\Delta}_M]$. Clearly, $\boldsymbol{\Delta}\in\mathcal{RH}_\infty$. Define the uncertainty set

$$\mathcal{D}_r:=\{\boldsymbol{\Delta}\in\mathcal{RH}_\infty\ :\ \|\boldsymbol{r}^{-1}\boldsymbol{\Delta}\|_\infty<1\}\tag{1}$$

and the family of plants

$$\mathcal{P}:=\{\boldsymbol{g}_p\ :\ \boldsymbol{\Delta}\in\mathcal{D}_r\}\ ,\tag{2}$$

where $\boldsymbol{r},\boldsymbol{r}^{-1}\in\mathcal{RH}_\infty$ characterizes the size of the uncertainty in the coprime factors at each frequency ω.

Consider the feedback system in Figure 1. The sensor noise $n(t)$ affects only the measurements of the output signal. We model the effect of the noise n on the measurements as additive uncertainty in the complex plane. Hence the corresponding uncertainty set is defined as follows: $\mathcal{W}:=\{w_a\in\mathbb{C}\ :\ |w_a|<L_a\}$. Define the transfer function from v to y_m in Figure 1 as $\boldsymbol{t}:=\frac{\boldsymbol{k}_2\boldsymbol{g}_p}{1+\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{g}_p}$.

Problem 1 *Given nonzero closed-loop noisy frequency-response data $\{\phi_i\}_{i=1}^N$ at $\omega_1,\ldots,\omega_N$, do there exist $\boldsymbol{\Delta}\in\mathcal{D}_r$ and complex noises $\{w_{ai}\}_{i=1}^N$ in $\mathcal{W}$ such that the closed-loop system of Figure 1 is internally stable and $\boldsymbol{t}(j\omega_i)+w_{ai}=\phi_i$, $i=1,\ldots,N$?*

Fix a measurement frequency ω and let ϕ be the complex measurement. The block diagram of Figure 2 represents a general model/data consistency equation at frequency ω.

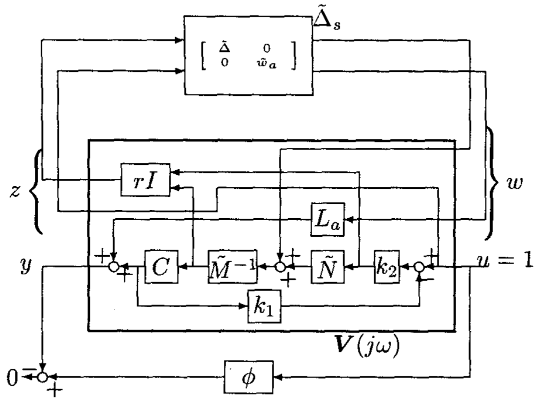

Figure 2: Block diagram of consistency equation.

Assumptions

(A1) $\phi \neq 0$ (nonzero complex measurement).

(A2) $d = \boldsymbol{d}(j\omega) := (1 + \boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{g})(j\omega) \neq 0,\ \forall\omega \in \mathbb{R}$.

(A3) $(\phi - V_{22}) \neq 0$, where V_{22} is the map $u \mapsto y$ in Figure 2 for $\tilde{\Delta}_s = 0$.

(A4) $k_1 \neq 0$ and $k_2 \neq 0$.

(A5) $\forall\boldsymbol{\Delta} \in \mathcal{D}_r$, the pair $(C,\ \tilde{\boldsymbol{M}}_p)$ is right-coprime.

Assumptions (A1) and (A3) should hold generically in practice. Assumption (A5) is required because otherwise robust internal stability could not be achieved. The generalized consistency equation at frequency ω illustrated in Figure 2 is the following (the ˜ mark for perturbations denotes normalization):

$$\phi - \left[1 + C(\tilde{M} + r\tilde{\Delta}_M)^{-1}(\tilde{N} + r\tilde{\Delta}_N)k_1k_2\right]^{-1} \cdot C(\tilde{M} + r\tilde{\Delta}_M)^{-1}(\tilde{N} + r\tilde{\Delta}_N)k_2 - L_a\tilde{w}_a = 0. \quad (3)$$

Using LFT notation, (4) becomes

$$\phi - \mathcal{F}_U\left[\boldsymbol{V}(j\omega), \tilde{\Delta}_s\right] = 0, \quad (4)$$

where

$$\tilde{\Delta}_s := \begin{bmatrix} \tilde{\Delta} & 0 \\ 0 & \tilde{w}_a \end{bmatrix}, \text{ and } \boldsymbol{V} := \begin{bmatrix} \boldsymbol{V}_{11} & \boldsymbol{V}_{12} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{bmatrix},$$

$$\boldsymbol{V}_{11} := \begin{bmatrix} -\boldsymbol{d}^{-1}\boldsymbol{r}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{C}\tilde{\boldsymbol{M}}^{-1} & 0 \\ \boldsymbol{r}(I + \tilde{\boldsymbol{M}}^{-1}\tilde{\boldsymbol{N}}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{C})^{-1}\tilde{\boldsymbol{M}}^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\boldsymbol{V}_{12} := \begin{bmatrix} \boldsymbol{r}\boldsymbol{d}^{-1}\boldsymbol{k}_2 \\ \boldsymbol{r}\boldsymbol{d}^{-1}\boldsymbol{k}_2\tilde{\boldsymbol{M}}^{-1}\tilde{\boldsymbol{N}} \\ 1 \end{bmatrix},$$

$$\boldsymbol{V}_{21} := \begin{bmatrix} \boldsymbol{d}^{-1}\boldsymbol{C}\tilde{\boldsymbol{M}}^{-1} & L_a \end{bmatrix},\ \boldsymbol{V}_{22} := \boldsymbol{d}^{-1}\boldsymbol{k}_2\boldsymbol{C}\tilde{\boldsymbol{M}}^{-1}\tilde{\boldsymbol{N}}.$$

Let $V := \boldsymbol{V}(j\omega)$, $V_{ij} := \boldsymbol{V}_{ij}(j\omega)$, and likewise for the other transfer functions and matrices in the definition of $\boldsymbol{V}$ above. Thus the general consistency problem at frequency ω can be stated as follows.

Problem 2 *Do there exist $\tilde{\Delta} \in \mathcal{B}\mathbb{C}^{n\times(n+1)}$ and a noise $w_a \in \mathcal{W}$ such that $I - V_{11}\tilde{\Delta}$ is nonsingular and $\phi - \mathcal{F}_U(V, \tilde{\Delta}_s) = 0$?*

The key idea for solving this problem is to write an equivalent consistency equation that has the interpretation of a feedback interconnection of ϕ^{-1} and $\mathcal{F}_U(V, \tilde{\Delta}_s)$: $1 - \phi^{-1}\mathcal{F}_U(V, \tilde{\Delta}_s) = 0$. Then consistency becomes analogous to instability of a feedback system, and the structured singular value μ can be used to derive consistency results. Based on this idea, we have the following lemma providing a solution to Problem 2 (See [1] for a proof). Let the structured uncertainty set be defined as $\Gamma := \left\{\tilde{\Delta}_s = \begin{bmatrix} \tilde{\Delta} & 0 \\ 0 & \tilde{w}_a \end{bmatrix} : \tilde{\Delta} \in \mathbb{C}^{n\times(n+1)},\ \tilde{w}_a \in \mathbb{C}\right\}$.

Lemma 1 *$\exists\tilde{\Delta}_s \in \mathcal{B}\Gamma$ such that $\phi - \mathcal{F}_U(V, \tilde{\Delta}_s) = 0$ and $I - V_{11}\tilde{\Delta}_s$ is nonsingular iff $\mu_\Gamma\left[\mathcal{F}_L(V, \phi^{-1})\right] > 1$.*

Our main consistency result gives a necessary condition for a positive answer to Problem 1 (see proof in [1]).

Theorem 1 *The noisy closed-loop SISO model/data consistency problem (Problem 1) has a positive answer only if $\mu_\Gamma\left\{\mathcal{F}_L\left[\boldsymbol{V}(j\omega_i), \phi_i^{-1}\right]\right\} > 1$ for $i = 1, \ldots, N$.*

References

[1] B. Boulet. *Modeling and Robust Control of Large Flexible Space Structures.* PhD thesis, Department of Electrical and Computer Engineering, University of Toronto, 1996.

[2] B. Boulet and B. A. Francis. Consistency of open-loop experimental frequency-response data with coprime factor plant models. *IEEE Transactions on Automatic Control*, 43(12):1680–1691, December 1998.

[3] B. Boulet, B. A. Francis, P. C. Hughes, and T. Hong. Uncertainty modeling and experiments in $\mathcal{H}_\infty$ control of large flexible space structures. *IEEE Transactions on Control Systems Technology*, 5(5):504–519, September 1997.

[4] J. Chen. Frequency-domain tests for validation of linear fractional uncertain models. *IEEE Transactions on Automatic Control*, 42(6):748–760, June 1997.

[5] R. S. Smith and J. C. Doyle. Model validation: A connection between robust control and identification. *IEEE Transactions on Automatic Control*, 37(7):942–952, July 1992.